# Statistical Orbit Determination

Lecture 2 – Orbital Mechanics Review A

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# Recap

- Lecture 1 – OD Concepts posted here
  - State estimation (what orbit determination is)
  - Linearization and state transition matrix
- Problem due Wed, 13 June
  - Quick review at beginning of this lecture
- Questions
  - Post them to lecture page

- Additional notes
  - Website revamp

# Agenda

- Problem review
- Two body problem
  - Gravitational force
  - Relative motion
  - n-body problem
- Orbital elements and $\vec{r}/\vec{v}$
  - Conic sections
  - Coordinate systems
- Perturbing accelerations
  - Conservative
  - Gravitational models
- Practice problem

# Problem review

| Time | 0.0 | 1.0 | 2.0 | 3.0 | 4.0 |
|---|---|---|---|---|---|
| Range, $\rho$ | 7.000000000 | 8.003905970 | 8.944271910 | 9.801147892 | 10.630145813 |
| Calculated Range, $\hat{\rho}$ | 9.013878189 | 9.73203473 | 10.6004717 | 11.5815586 | 12.66688596 |
| $X_0$ | 1.5 | 3.7 | 5.9 | 8.1 | 10.3 |
| $Y_0$ | 10.0 | 10.35 | 10.4 | 10.15 | 9.6 |
| $\dot{X}_0$ | 2.2 | 2.2 | 2.2 | 2.2 | 2.2 |
| $\dot{Y}_0$ | 0.5 | 0.2 | -0.1 | -0.4 | -0.7 |
| $g$ | 0.3 | 0.3 | 0.3 | 0.3 | 0.3 |
| $X_s$ | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 |
| $Y_s$ | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 |

# Two body problem – Gravitational Force (1/2)

- Newton's Law of Universal Gravitation

$$\vec{F} = -\frac{GmM}{r^2}\frac{\vec{r}}{r}$$

- Assumptions
  - Two point masses/bodies are spherically symmetric
  - Gravitational force propagates instantaneously (No relativistic effects)
  - Only forces in the system are the gravitational attractions
- Gravitational constant, $G = 6.6742 \times 10^{-20} \frac{\text{km}^3}{\text{kg}-\text{s}^2}$
- Earth's estimated mass, $M_\oplus = 5.9722 \times 10^{24}\ \text{kg}$
- Gravitational parameter, $\mu = M_\oplus G = 3.98 \times 10^5 \frac{\text{km}^3}{\text{s}^2}$

# Two body problem – Gravitational Force (2/2)

- Gravitational acceleration, $g$

$$g \equiv \left\| \frac{\vec{F}}{m} \right\| = -\frac{GM}{r^2} = -\frac{\mu}{r^2}$$

- Field acceleration on surface of uniformly dense sphere gives $g_0$
  - Earth's radius is $6378 \text{ km}$

$$g_0 = -\frac{\mu}{r^2} = -\left( \frac{3.98 \times 10^5 \frac{\text{km}^3}{\text{s}^2}}{(6378 \text{ km})^2} \right) \cong -9.81 \text{ m/s}^2$$

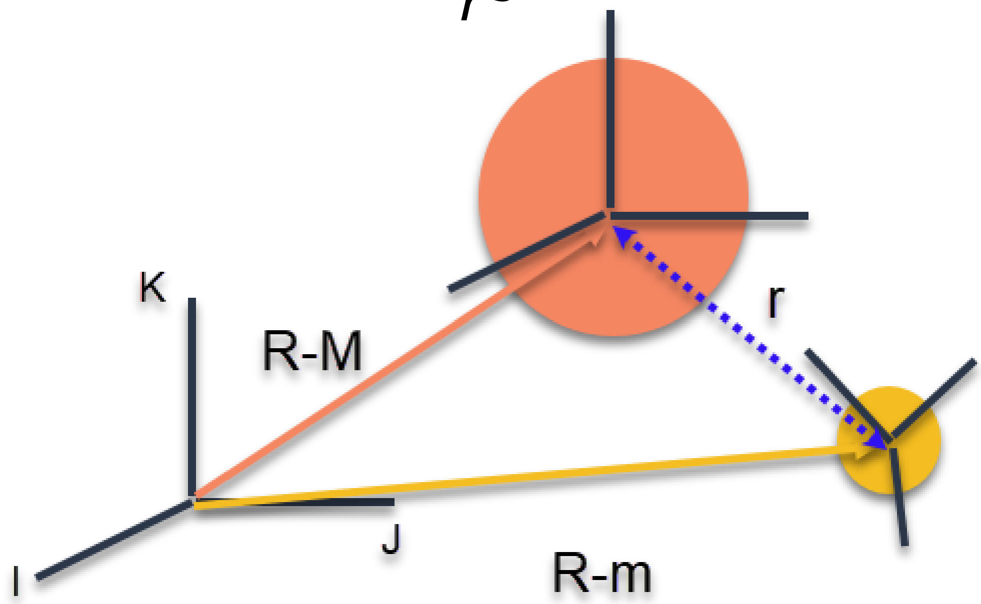
# Two body problem – Relative motion

- When $M \gg m$

$$\mu \cong G(M+m) = GM$$

- Acceleration of point mass, $m$

$$\ddot{\vec{r}} + \frac{\mu}{r^3}\vec{r} = 0$$



# Two body problem – n-body problem

- Assume a system of n-bodies
  - $(m_1, m_2, \dots, m_n)$ and we want to examine $m_i$
  - Sum the vectors acting on $m_i$

$$\vec{F}_g = -G m_i \sum_{j=1, j\neq i}^{n} \frac{m_j}{r_{ij}^3} \vec{r}_{ji}$$

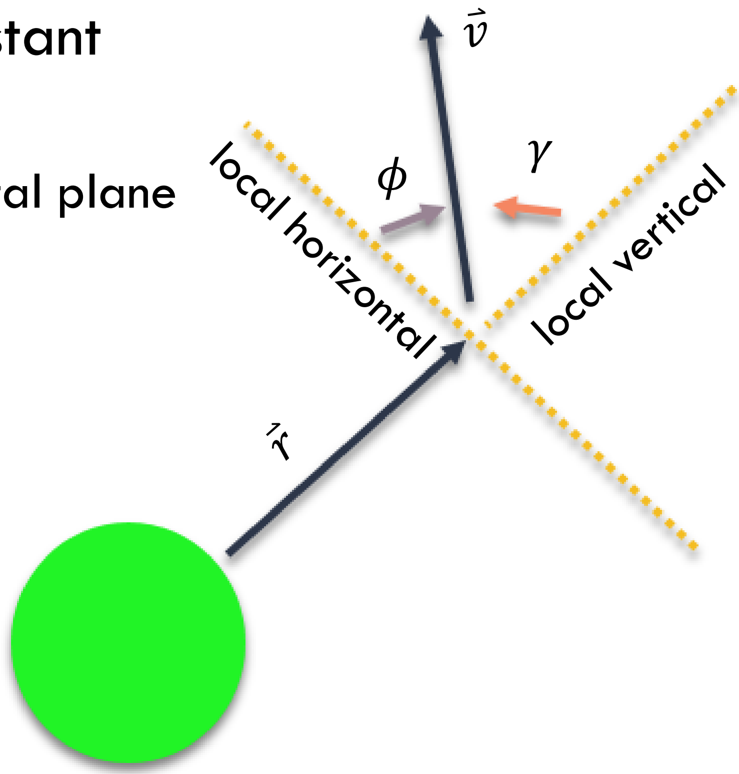
- Adding all other forces

$$\ddot{\vec{r}}_i = \frac{\vec{F}_g + \vec{F}_{\text{other}}}{m_i} - \dot{\vec{r}}_i \left(\frac{\dot{m}_i}{m_i}\right)$$

- Second order, nonlinear, vector, differential equation of motion
  - Not solved in present form

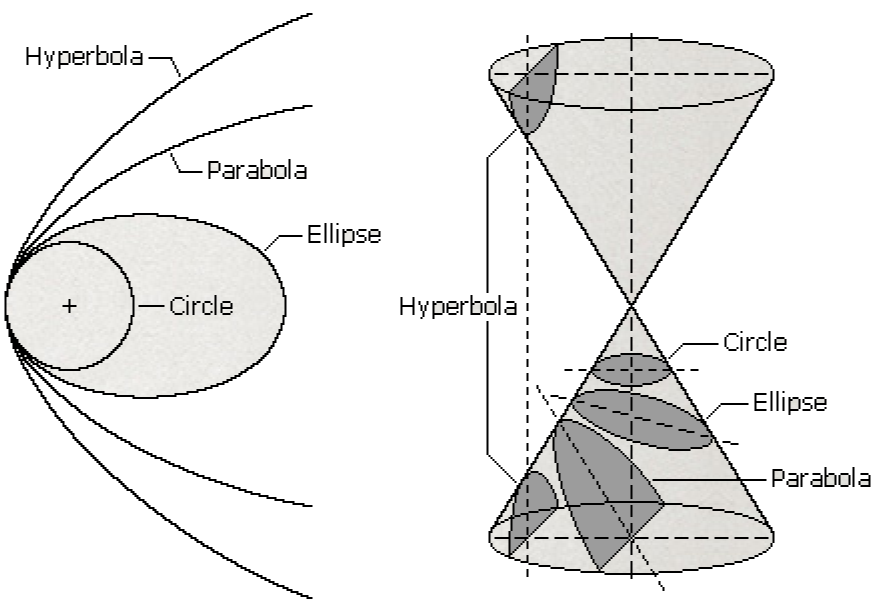
# Orbital elements and $\vec{r}/\vec{v}$ – Conic Sections (1/4)

- Gravitational field is conservative
  - Object moving under influence of $g$ only exchanges KE for PE
  - Specific mechanical energy is constant for each orbit
  - $\varepsilon = \frac{v^2}{2} - \frac{\mu}{r}$
- Specific angular momentum is constant
  - $\vec{h} = \vec{r} \times \vec{v}$
  - Position and velocity act only in orbital plane
  - Flight path angle, $\phi = \mathrm{acos}\frac{h}{rv}$
  - Zenith angle, $\gamma = \mathrm{asin}\frac{h}{rv}$



# Orbital elements and $\vec{r}/\vec{v}$ – Conic Sections (2/4)

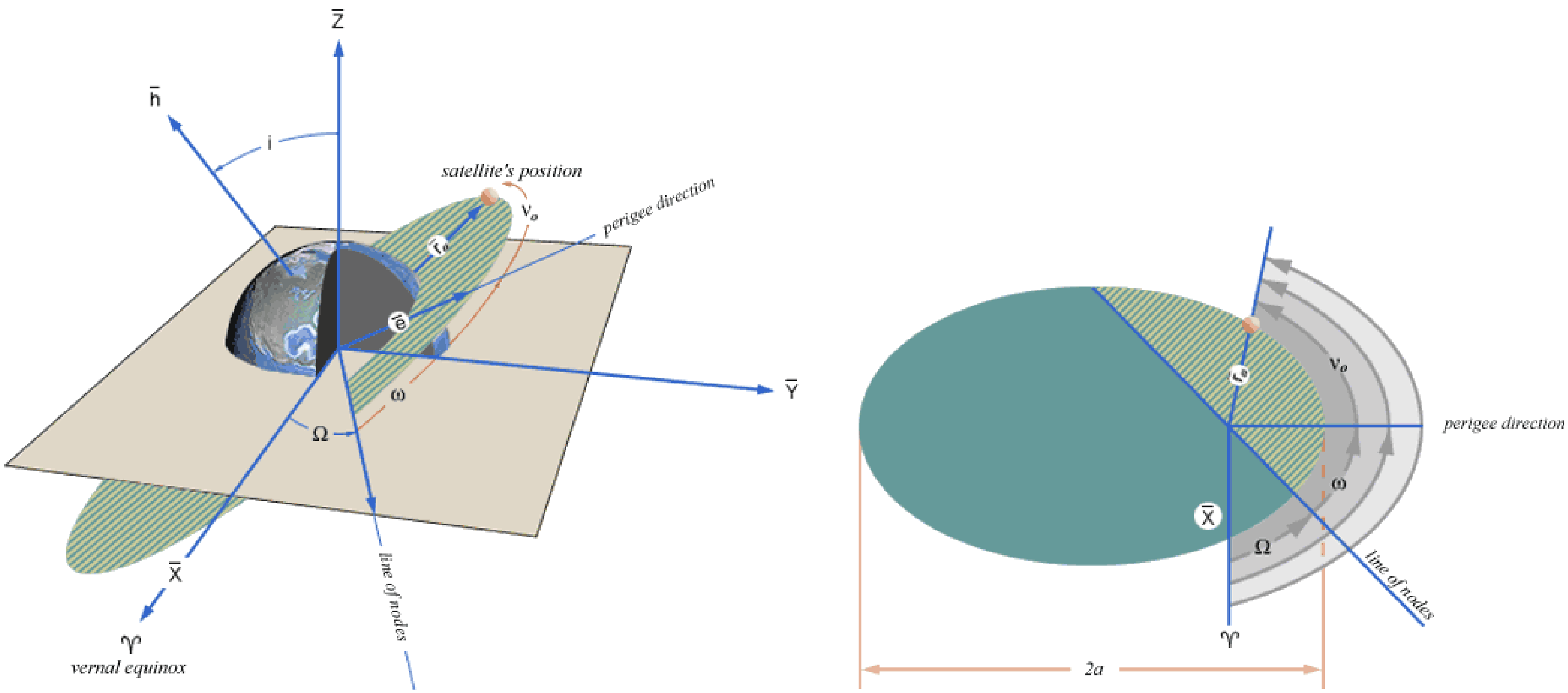
- Conic sections, family of curves, only trajectories for orbit
  - Circle, $e = 0$, ellipse, $0 < e < 1$, parabola, $e = 1$, hyperbola, $e > 1$
  - In polar coordinates, $r = \frac{p}{1+e\cos\nu}$
    - Semilatus rectum, $p = a\,(1 - e^2) = h^2/\mu$
    - Eccentricity, $e = \sqrt{1 + \frac{2\varepsilon h^2}{\mu^2}}$
  - The focus of the conic orbit is the center of the central body



# Orbital elements and $\vec{r}/\vec{v}$ – Conic Sections (3/4)

- Six parameters needed to describe an orbit
  - Classical orbital elements
    - $a$, semi-major axis
    - $e$, eccentricity
    - $i$, inclination
    - $\Omega$, longitude of the ascending node
    - $\omega$, argument of periapsis
    - T, time of periapsis passage

# Orbital elements and $\vec{r}/\vec{v}$ – Conic Sections (4/4)



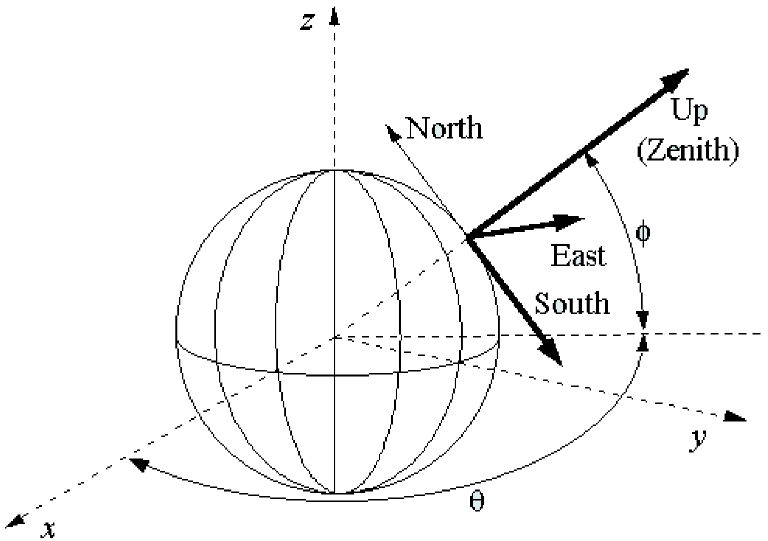

a - defines the size of the orbit
e - defines the shape of the orbit
i - defines the orientation of the orbit with respect to the Earth's equator.
ω - defines where the low point, perigee, of the orbit is with respect to the Earth's surface.
Ω - defines the location of the ascending and descending orbit locations with respect to the Earth's equatorial plane.
ν - defines where the satellite is within the orbit with respect to perigee.

# Orbital elements and $\vec{r}/\vec{v}$ – Coordinate Systems (1/2)

- Heliocentric-Ecliptic
  - Origin at center of Sun
  - Fundamental plane is ecliptic or Earth's orbital plane about the Sun
  - Used as inertial reference frame when defined by an epoch
- Geocentric-Equatorial
  - Origin at center of Earth
  - Fundamental plane is equatorial plane
  - Not fixed to Earth
- Right Ascension-Declination
  - Origin at center of Earth or point on surface of Earth
  - Fundamental plane is equatorial plane fixed to the celestial sphere
  - Star positions known accurately, satellite in background

# Orbital elements and $\vec{r}/\vec{v}$ – Coordinate Systems (2/2)

- Topocentric-Horizon Coordinate System
  - Fundamental plane is horizon
  - X points South, Y points East, and Z points up



# Perturbing accelerations – Conservative (1/2)

- Acceleration of satellite with perturbing accelerations

$$\ddot{\vec{r}} + \frac{\mu}{r^3}\vec{r} = \ddot{\vec{r}}_p$$

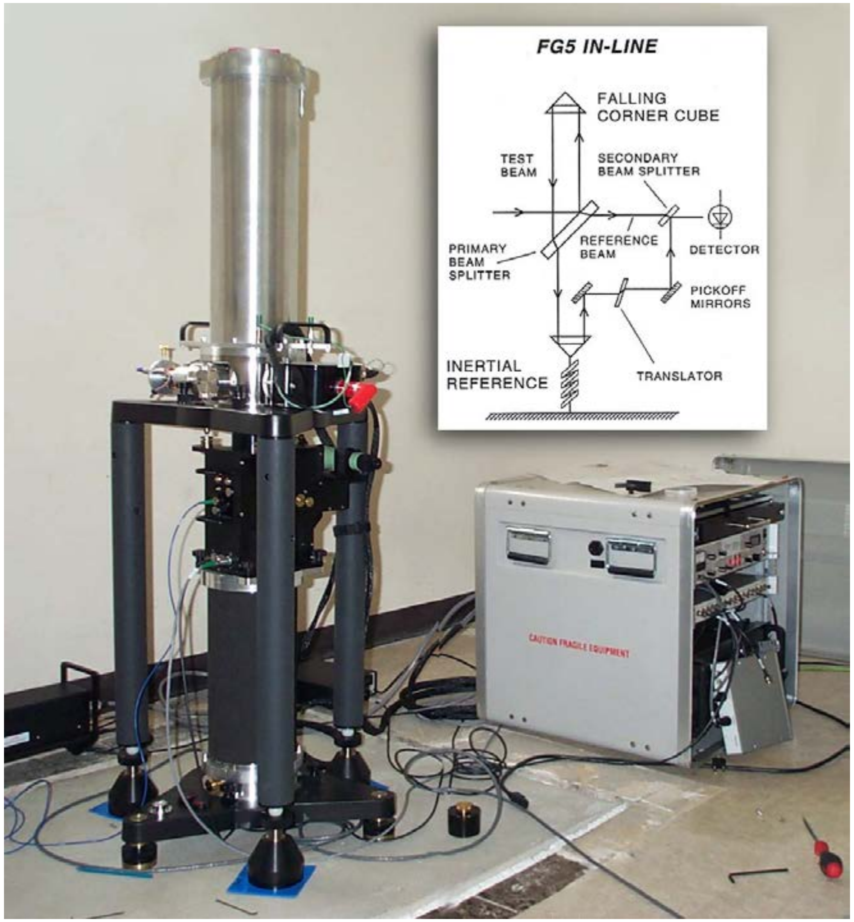
- Perturbations are conservative if only a function of position
  - Satellite does not lose nor gain mechanical energy
  - Exchanges energy between kinetic energy and potential energy
  - Specific mechanical energy is unique for each orbit
- Examples of non-conservative perturbations (changes to $\vec{r}, \vec{v}$ )
  - Atmospheric drag
  - Outgassing
  - Tidal effects

# Perturbing accelerations – Conservative (2/2)

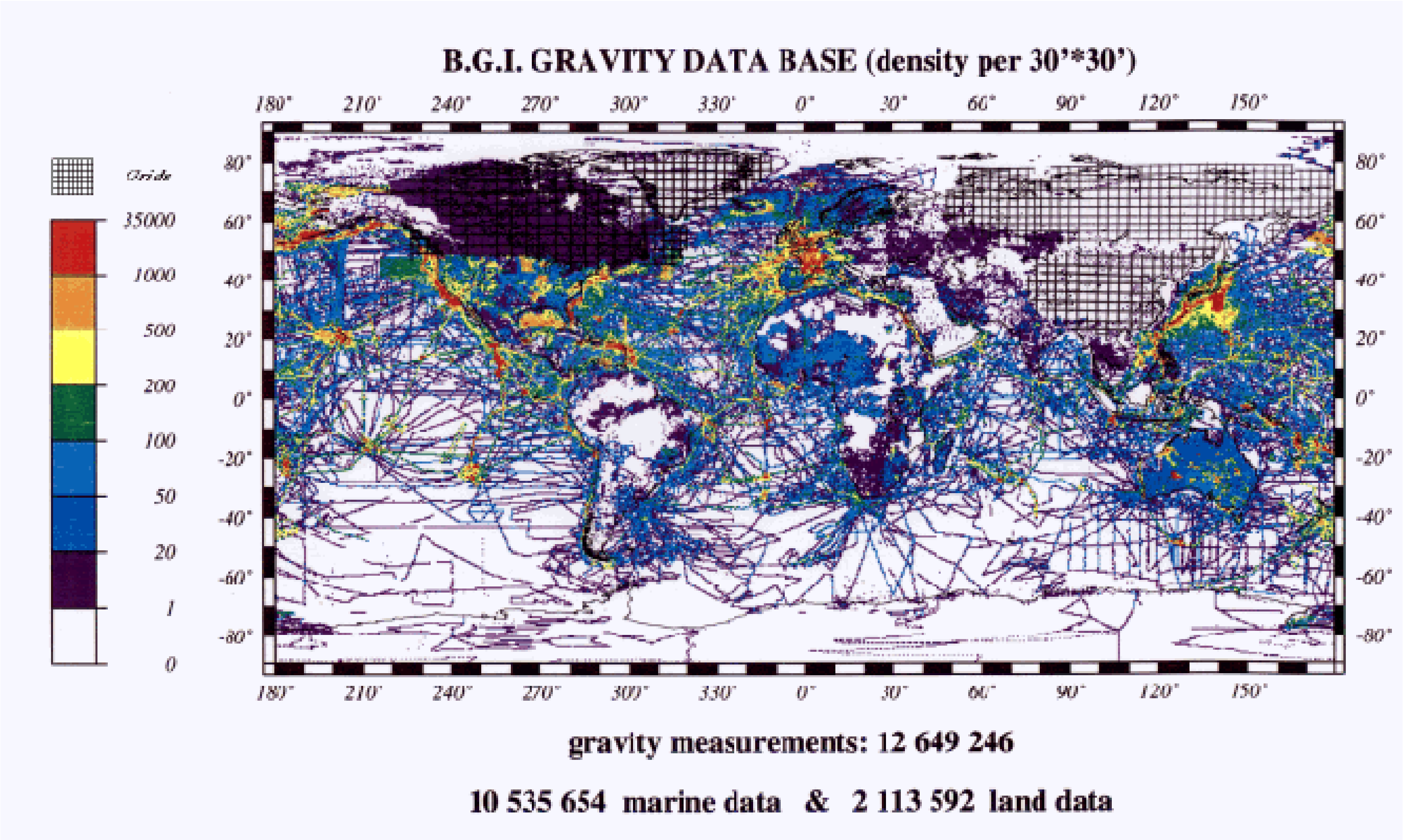
- Examples of conservative perturbations
  - N-body (celestial body) attractions
  - Nonspherical celestial bodies
  - Solar-radiation pressure
- Focus on the gravitational field effects
  - Nonspherical celestial bodies
  - Tidal effects
  - N-body attractions

# Perturbing accelerations – Gravitational Models (1/6)

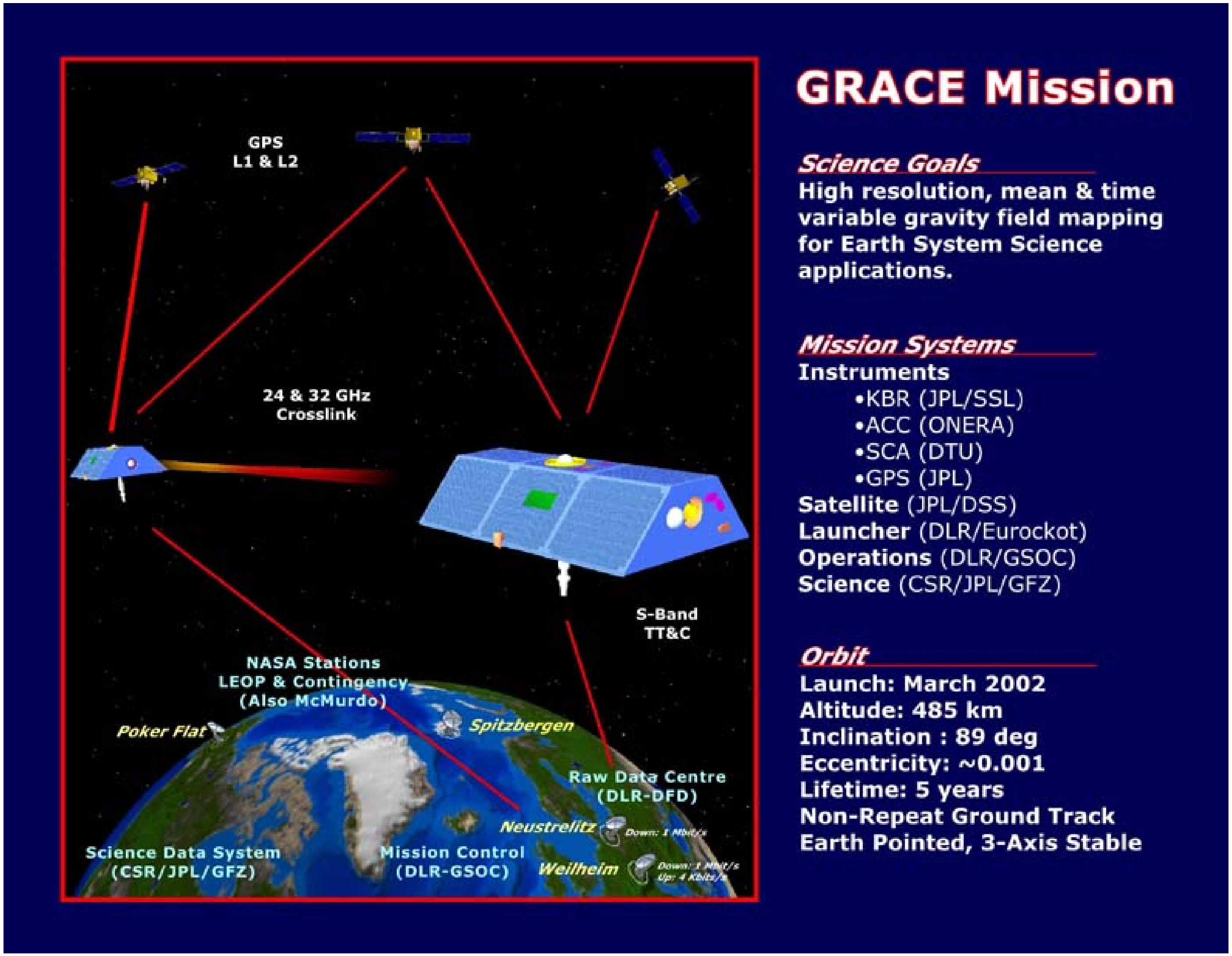
- Terrestrial Measurements



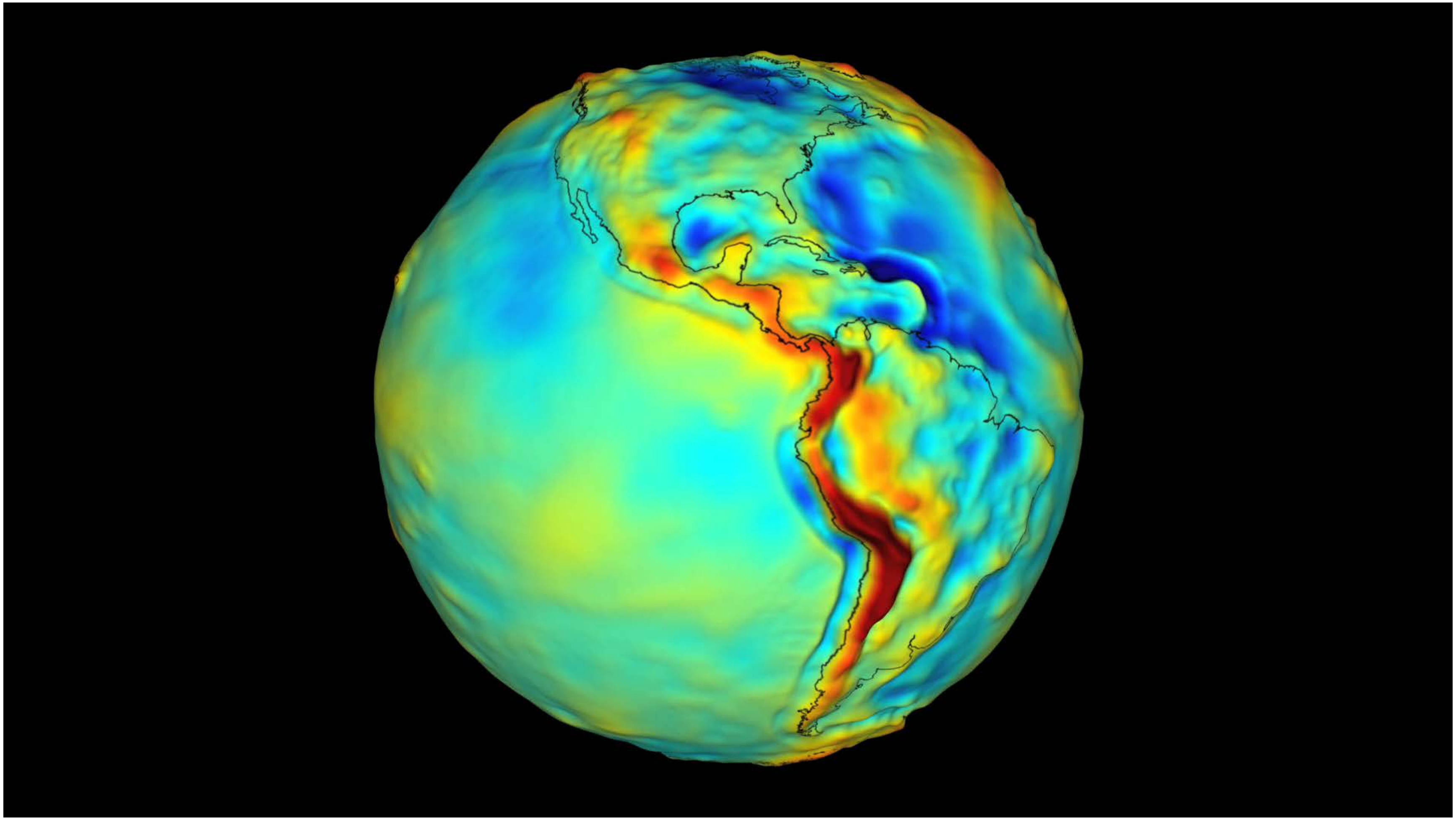

# Perturbing accelerations – Gravitational Models (2/6)



# Perturbing accelerations – Gravitational Models (3/6)

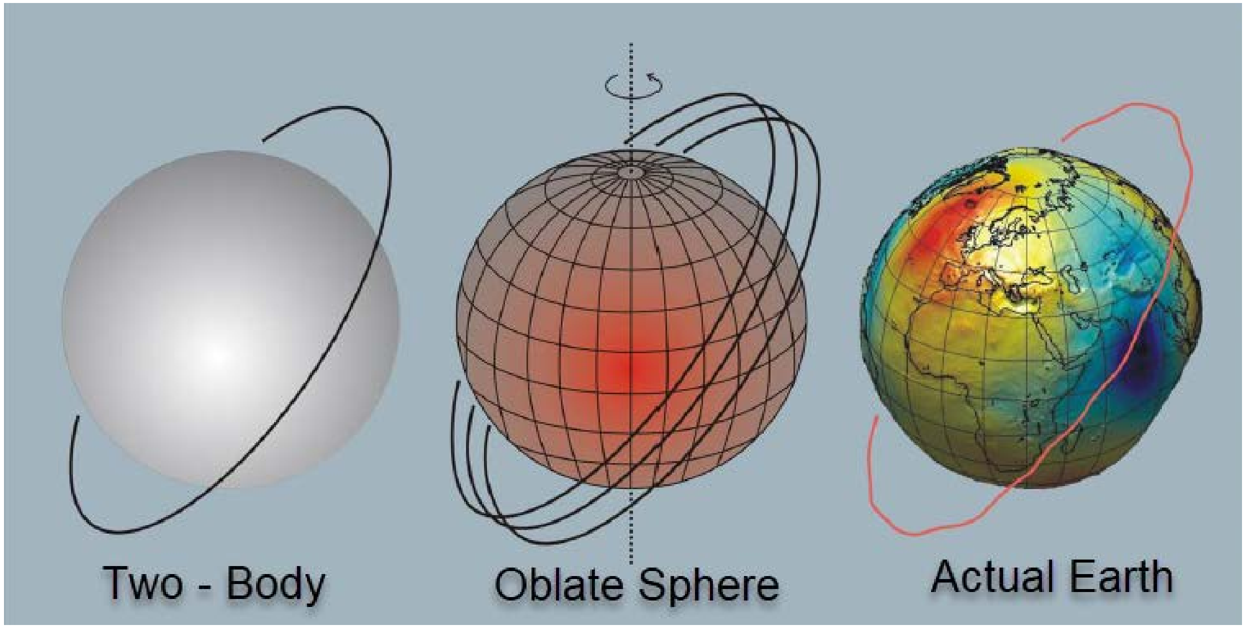


# Perturbing accelerations – Gravitational Models (4/6)



# Perturbing accelerations – Gravitational Models (5/6)

- Earth's Oblateness $(J_{2,0})$
  - Bulging at the equator
  - ~400 times larger than the next term
  - When included in satellite orbits maintains reasonable accuracy



# Perturbing accelerations – Gravitational Models (6/6)

- Earth's bulge at equator pulls satellite down faster
  - Exerts a force component toward the equator
- Satellite reaches equator short of point for spherical Earth
  - East-bound satellite goes west
  - West-bound satellite goes east

$$\dot{\Omega} = -\frac{9.9358}{(1-e^2)^2}\left(\frac{r_{eq}}{r_{eq}+\bar{h}}\right)^{3.5} \cos i \;\; [\text{deg/mean solar day}]$$

- Secular motion of perigee too
  - Force is no longer proportional to inverse square radius

$$\dot{\omega} = \frac{9.9358}{(1-e^2)^2}\left(\frac{r_{eq}}{r_{eq}+\bar{h}}\right)^{3.5}\left(2 - \frac{5}{2}\sin^2 i\right) \; [\text{deg/mean solar day}]$$

Practice problem: Gibbsian method

# LOST IN SPACE