Statistical Orbit Determination

Lecture 1 – Orbit Determination Concepts
Presenter: Christopher R. Simpson
Introduction

• Welcome!
  – Class Schedule: MWF
  – Syllabus available here

• Working through material
  – If you have a question, leave it on the YouTube page
  – I would like to encourage discussion among those of you working through the material
Agenda

• Overview
  – Inherent characteristics of OD problem
• Dynamic System
  – Dynamic state estimation
    – Uniform gravitational field example
  – Observations
    – Non-linear functions of state variables
  – Linearization
  – State transition matrix
• Example Problem
Overview – Obtain Knowledge

• Orbit Determination is
  – The process by which we obtain knowledge of satellite motion
    – “The problem of determining the best estimate of the state of a spacecraft whose initial state is unknown, from observations influenced by random and systematic errors, using a mathematical model that is not exact, is ... the problem of state estimation,” or orbit determination [1]

• Let the astronomers focus on the heavenly bodies
  – We will focus on artificial (noncelestial) satellites
Overview – Characteristics

• Dynamic state estimation
  – Set of parameters required to predict future motion
  – Initially, just position and velocity vectors
  – Later, we will include dynamic and measurement model parameters

• Best estimate
  – Truth of observation and reported state
Overview – Estimates

• “Generating an ephemeris,” is predicting the state of a vehicle
  – An ephemeris is a table of position and velocity as function of time

• Predicted values differ from true values for 2 reasons

• 1. Inaccuracies in estimated state vector
  – Approximations in model and method of orbit propagation
  – Errors in observations
  – Errors in computational procedures

• 2. Errors in numerical integration
  – Caused by dynamical model and roundoff errors and truncation
Dynamic System – Estimates (1/2)

• Inaccuracies in estimate and errors in numerical integration

• Review probability, statistics and matrix theory
  – See Appendix A and B or notes on the appendices

• Will use a uniform gravitational field to illustrate OD process
  – True trajectory, \( X \)
  – Nominal trajectory, \( X^* \)
  – (Best) Estimate trajectory, \( \hat{X} \)

• \( X \) is our state vector
  – Observations
  – Location of Observations
  – Estimate method
Dynamic System – Estimates (2/2)
Dynamic System – Observing (1/3)

• Updating $X$ is inherently linear
  - $X = A[\dot{X}]$
  - Ex: $X(t) = X_0 + \dot{X}_0 t$

• Cannot observe individual state components directly

• Observations are non-linear
  - Range, $\rho$, range-rate, $\dot{\rho}$, elevation, $\theta$, elevation-rate, $\dot{\theta}$, etc.
  - $\rho(t) = \sqrt{(X(t) - X_s)^2 + (Y(t) - Y_s)^2}$
  - $\tan(\theta(t)) = (Y(t) - Y_s)/(X(t) - X_s)$
  - $\dot{\rho}(t) = \frac{1}{\rho} \left[ (X(t) - X_s)(\dot{X}(t) - \dot{X}_s) + (Y(t) - Y_s)(\dot{Y}(t) - \dot{Y}_s) \right]$
  - $\dot{\theta}(t) = \frac{1}{\rho^2} \left[ (X(t) - X_s)(\dot{Y}(t) - \dot{Y}_s) - (\dot{X}(t) - \dot{X}_s)(Y(t) - Y_s) \right]$
Dynamic System – Observing (2/3)

• Transforming observations into the best estimate
  
  \[ J(X_0) \equiv Y - G(X_0, t) = 0 \]
  
• \( Y \) is our observations and \( G \) is a 4x1 nonlinear vector from the equations in the previous slide
  
• Best estimate
  
  – Solve \( J(X_0) \) using Newton-Raphson iteration
  
  – Iteration is repeated until \( \|X_{0}^{n+1} - X_{0}^{n}\| \leq \varepsilon \)

• Since equations are nonlinear, multiple solutions may exist
Dynamic System – Observing (3/3)

• Not all observation information may be available
  – Range and elevation but not range-rate nor elevation rate

• \((\rho, \theta)\) will be insufficient to determine \(X\)
  – Two observations at different times will be required
  – The set \((\rho_1, \theta_1, \rho_2, \theta_2)\) is sufficient

• Both approaches assume
  – Perfect knowledge of governing diff. eqns
  – Perfect observations

• In general many observations will be taken
  – This will decrease variance and eliminate random/systematic errors
• Best estimate is obtained by linearizing the problem
  – Expand equations of motion
  – Expand observation-state relationship about a reference trajectory

• Deviations from reference are determined for best-fit
  – Minimize variance to yield best agreement with observations
  – Generally based on least-squares criterion
Dynamic System – Linearization (2/4)

• Using the flat-earth example in the previous slide
  – Assume errors in initial position, velocity, and in $g$ values
  – $X = X^* + \delta X$ or the nominal plus the perturbed state
    – Perturbed means deviation (simplistically) in this case

• Write a state vector, subtracting the nominal
  – $\delta X = X - X^*$, $\delta X^T = [\delta X, \delta Y, \delta \dot{X}, \delta \dot{Y}, g]$
    – $\delta X = \delta X_0 + \delta \dot{X}_0 t$
    – $\delta Y = \delta Y_0 + \delta \dot{Y}_0 t - \delta g \left(\frac{t^2}{2}\right)$
    – $\delta \dot{X} = \delta \dot{X}_0$
    – $\delta \dot{Y} = \delta \dot{Y}_0 - \delta gt$
    – $\delta g = \delta g$
• Observations becoming linear
  
  – Observations linearized by expanding in Taylor series about $X^*$
  
  \[
  \rho \cong \rho^* + \left[ \frac{\delta \rho}{\delta X} \right]^* (X - X^*) + \left[ \frac{\delta \rho}{\delta Y} \right]^* (Y - Y^*) + \epsilon_\rho \\
  \theta \cong \theta^* + \left[ \frac{\delta \theta}{\delta X} \right]^* (X - X^*) + \left[ \frac{\delta \theta}{\delta Y} \right]^* (Y - Y^*) + \epsilon_\theta \\
  \dot{\theta} \cong 0 \\
  \dot{\rho} \cong 0
  \]

• Rewriting
  
  \[
  \delta \rho = \left[ \frac{\delta \rho}{\delta X} \right]^* \delta X + \left[ \frac{\delta \rho}{\delta Y} \right]^* \delta Y + \epsilon_\rho \\
  \delta \theta \cong \left[ \frac{\delta \theta}{\delta X} \right]^* \delta X + \left[ \frac{\delta \theta}{\delta Y} \right]^* \delta Y + \epsilon_\theta
  \]
Dynamic System – Linearization (4/4)

• Determining the deviation from the nominal trajectory
  – We have linearized the system
  – We can now use linear algebra to our advantage
    \[ y = \tilde{H}x + \varepsilon \]
  – \( y^T = [\delta \rho \ \delta \theta] \)
  – \( \tilde{H} \) is the mapping vector of partial derivatives with respect to \( X^* \)
  – \( x^T = [\delta X \ \delta Y \ \delta \dot{X} \ \delta \dot{Y} \ \delta g] \)
  – \( \varepsilon^T = [\varepsilon \rho \ \varepsilon \theta] \)

• Think about it: How can you use this? Why is this valuable?
• The state transition matrix ($\Phi$) updates the deviation vector
  
  \[ [\delta X] = \Phi(t, t_0)[\delta X_0] \]

  - In other words, $\Phi$ maps deviations in the state vector from one time to another

• Classical orbit determination the mapping is exact

• General orbit determination
  
  - State equations are nonlinear
  
  - $\Phi$ is the linear term in a Taylor series expansion of $X(t)$ at $t_0$
\( \Phi \) maps \( x_0 \) to \( x(t) \)

- \( y(t) = \left[ \tilde{H}(t) \Phi(t, t_0) \right] \left[ \Phi(t, t_0) x_0 \right] + \varepsilon \)

- **Given an arbitrary epoch, \( t_k \)**
  - We can determine the best estimate of \( x_k \) using the deviations
  - \( H \) will be a \( m \times n \) vector
    - \( n \) is the number of state variables
    - \( m \) is the number of observations
  - In general OD \( m \geq n \) is always satisfied
  - In classical OD \( m = n \)

- **We will cover how to use these extra observations later**
Practice Problem: Uniform Gravitational Field (2D)

SIMPLE DYNAMIC SYSTEM
Simple Dynamic System – Problem Definition

Given perturbed initial conditions, use a Newton iteration scheme to recover the exact initial conditions or conditions used to produce the observations provided. Assume the ground station coordinates are correct.

- Write a computer program that computes $\rho(t_i)$
  - 2D uniform gravity field
- Compute observations from given initial conditions
- Iterate and solve for the correct initial conditions
- $X_0 = 1.5, Y_0 = 10.0, \dot{X}_0 = 2.2, \dot{Y}_0 = 0.5, g = 0.3, X_s = Y_s = 1.0$
- $\rho(t = 0, 1, 2, 3, 4) =$
  $7.0, 8.00390597, 8.94427191, 9.801147892, 10.630145813$