Statistical Orbit Determination

Lecture 2 – Orbital Mechanics Review

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Recap

- Lecture 2 – Notes posted here
  - State estimation (what orbit determination is)
  - Linearization and state transition matrix
- Problem due Mon, 28 Jan
  - Quick review at beginning of this lecture
- Questions
  - Post them to lecture page
- Additional notes
  - Website revamp

Agenda

- Problem review
- Two body problem
  - Gravitational force
  - Relative motion
  - n-body problem
- Orbital elements and $\vec{r}/\vec{v}$
  - Conic sections
  - Coordinate systems
- Perturbing accelerations
  - Conservative
  - Gravitational models
- Practice problem

Problem review

Time	0.0	1.0	2.0	3.0	4.0
Range, ρ	7.000000000	8.003905970	8.944271910	9.801147892	10.630145813
Calculated Range, $\hat{\rho}$	9.013878189	9.73203473	10.6004717	11.5815586	12.66688596
X_0	1.5	3.7	5.9	8.1	10.3
Y_0	10.0	10.35	10.4	10.15	9.6
$\dot{X}_0$	2.2	2.2	2.2	2.2	2.2
$\dot{Y}_0$	0.5	0.2	-0.1	-0.4	-0.7
g	0.3	0.3	0.3	0.3	0.3
X_s	1.0	1.0	1.0	1.0	1.0
Y_s	1.0	1.0	1.0	1.0	1.0

Two body problem – Gravitational Force (1/2)

- Newton's Law of Universal Gravitation

$$\vec{F} = -\frac{GmM}{r^2}\frac{\vec{r}}{r}$$

- Assumptions
  - Two point masses/bodies are spherically symmetric
  - Gravitational force propagates instantaneously (No relativistic effects)
  - Only forces in the system are the gravitational attractions
- Gravitational constant, $G = 6.6742 \times 10^{-20}\,\frac{\mathrm{km}^3}{\mathrm{kg{-}s}^2}$
- Earth's estimated mass, $M_{\oplus} = 5.9722\ \times 10^{24}\ \mathrm{kg}$
- Gravitational parameter, $\mu = M_{\oplus}G = 3.98\ \times 10^{5}\,\frac{\mathrm{km}^3}{\mathrm{s}^2}$

Two body problem – Gravitational Force (2/2)

- Gravitational acceleration, g

$$g \equiv \left\| \frac{\vec{F}}{m} \right\| = -\frac{GM}{r^2} = -\frac{\mu}{r^2}$$

- Field acceleration on surface of uniformly dense sphere gives g_0
  - Earth's radius is $6378\ \mathrm{km}$

$$g_0 = -\frac{\mu}{r^2} = -\left(\frac{3.98\ \times 10^5\,\frac{\mathrm{km}^3}{\mathrm{s}^2}}{(6378\ \mathrm{km})^2} \right) \cong -9.81\ \mathrm{m/s^2}$$

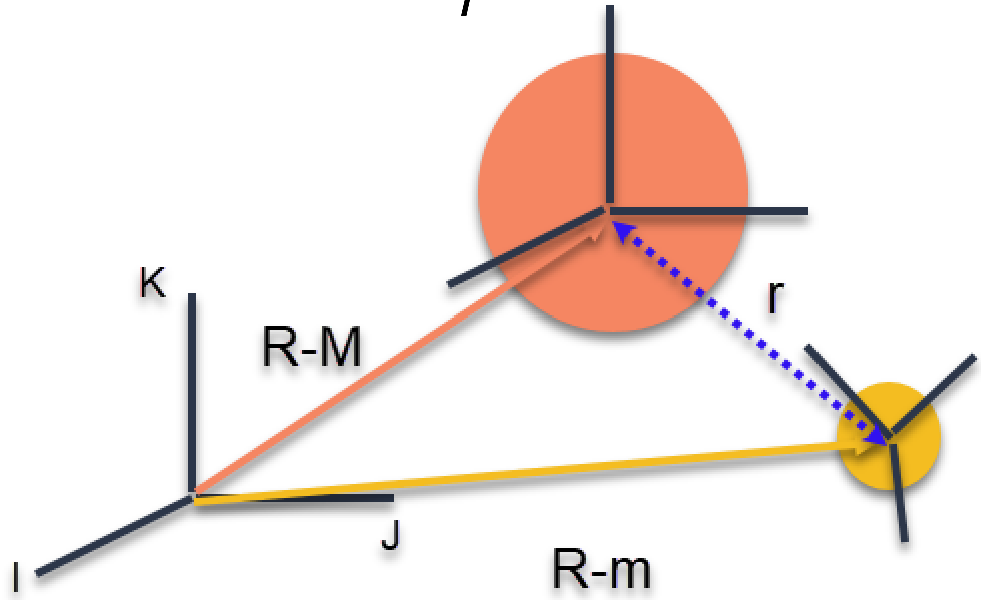
Two body problem – Relative motion

- When $M \gg m$

$$\mu \cong G(M+m) = GM$$

- Acceleration of point mass, m

$$\ddot{\vec{r}} + \frac{\mu}{r^3}\,\vec{r} = 0$$



Two body problem – n-body problem

- Assume a system of n-bodies
  - $(m_1, m_2, \ldots, m_n)$ and we want to examine m_i
  - Sum the vectors acting on m_i

$$\vec{F}_g = -Gm_i \sum_{j=1, j\neq i}^{n} \frac{m_j}{r_{ij}^3} \vec{r}_{ji}$$

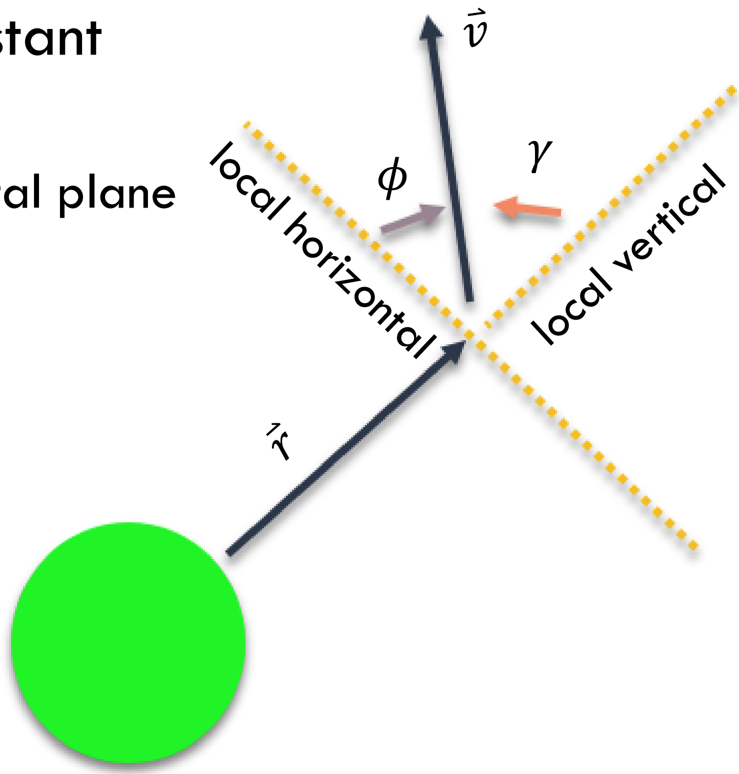
- Adding all other forces

$$\ddot{\vec{r}}_i = \frac{\vec{F}_g + \vec{F}_{\text{other}}}{m_i} - \dot{\vec{r}}_i \left(\frac{\dot{m}_i}{m_i}\right)$$

- Second order, nonlinear, vector, differential equation of motion
  - Not solved in present form

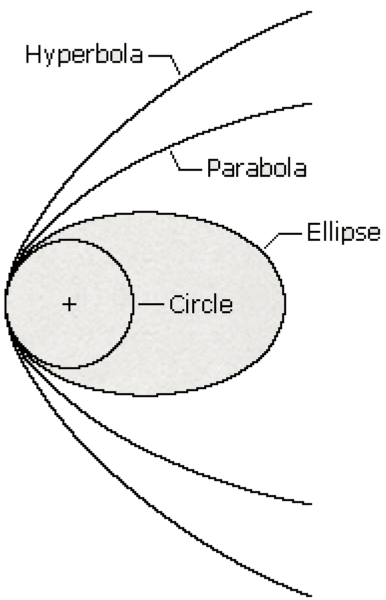
Orbital elements and $\vec{r}/\vec{v}$ – Conic Sections (1/4)

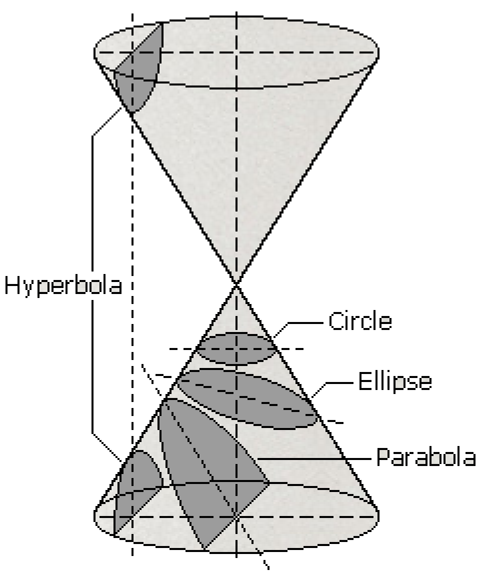
- Gravitational field is conservative
  - Object moving under influence of g only exchanges KE for PE
  - Specific mechanical energy is constant for each orbit
  - $\varepsilon = \frac{v^2}{2} - \frac{\mu}{r}$
- Specific angular momentum is constant
  - $\vec{h} = \vec{r} \times \vec{v}$
  - Position and velocity act only in orbital plane
  - Flight path angle, $\phi = \mathrm{acos}\frac{h}{rv}$
  - Zenith angle, $\gamma = \mathrm{asin}\frac{h}{rv}$



Orbital elements and $\vec{r}/\vec{v}$ – Conic Sections (2/4)

- Conic sections, family of curves, only trajectories for orbit
  - Circle, $e = 0$, ellipse, $0 < e < 1$, parabola, $e = 1$, hyperbola, $e > 1$
  - In polar coordinates, $r = \dfrac{p}{1+e\cos\nu}$
    - Semilatus rectum, $p = a\,(1 - e^2) = h^2/\mu$
    - Eccentricity, $e = \sqrt{1 + \dfrac{2\varepsilon h^2}{\mu^2}}$
  - The focus of the conic orbit is the center of the central body

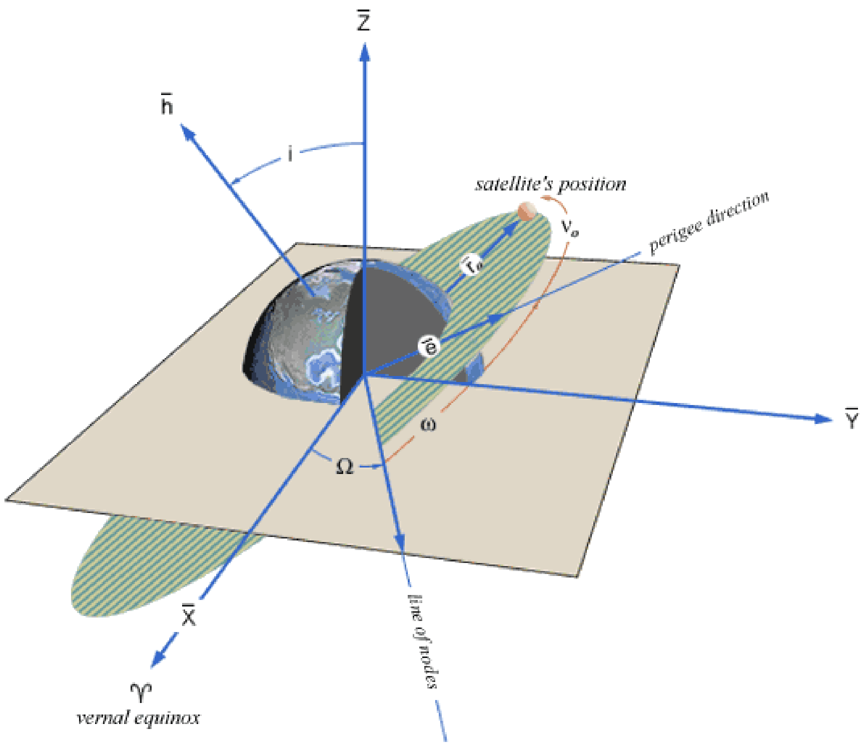


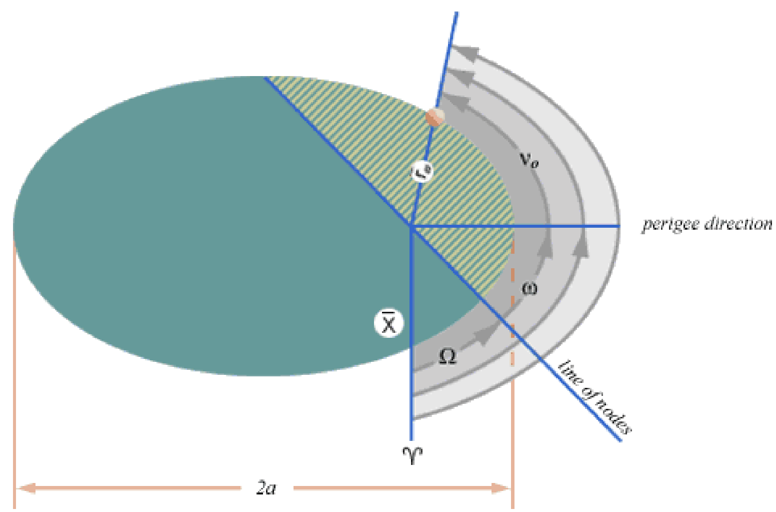


Orbital elements and $\vec{r}/\vec{v}$ – Conic Sections (3/4)

- Six parameters needed to describe an orbit
  - Classical orbital elements
    - a, semi-major axis
    - e, eccentricity
    - i, inclination
    - Ω, longitude of the ascending node
    - ω, argument of periapsis
    - T, time of periapsis passage

Orbital elements and $\vec{r}/\vec{v}$ – Conic Sections (4/4)





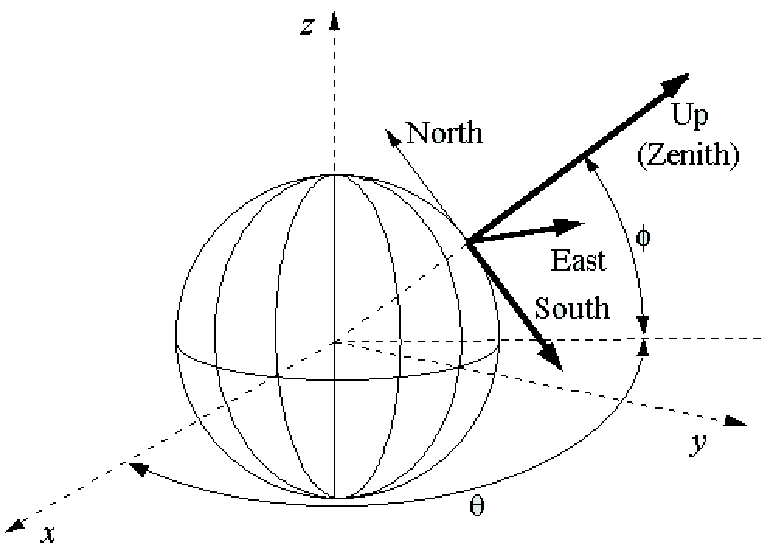
- a - defines the size of the orbit
- e - defines the shape of the orbit
- i - defines the orientation of the orbit with respect to the Earth's equator.
- ω - defines where the low point, perigee, of the orbit is with respect to the Earth's surface.
- Ω - defines the location of the ascending and descending orbit locations with respect to the Earth's equatorial plane.
- ν - defines where the satellite is within the orbit with respect to perigee.

Orbital elements and $\vec{r}/\vec{v}$ – Coordinate Systems (1/2)

- Heliocentric-Ecliptic
  - Origin at center of Sun
  - Fundamental plane is ecliptic or Earth's orbital plane about the Sun
  - Used as inertial reference frame when defined by an epoch
- Geocentric-Equatorial
  - Origin at center of Earth
  - Fundamental plane is equatorial plane
  - Not fixed to Earth
- Right Ascension-Declination
  - Origin at center of Earth or point on surface of Earth
  - Fundamental plane is equatorial plane fixed to the celestial sphere
  - Star positions known accurately, satellite in background

Orbital elements and $\vec{r}/\vec{v}$ – Coordinate Systems (2/2)

- Topocentric-Horizon Coordinate System
  - Fundamental plane is horizon
  - X points South, Y points East, and Z points up



Perturbing accelerations – Conservative (1/2)

- Acceleration of satellite with perturbing accelerations

$$\ddot{\vec{r}} + \frac{\mu}{r^3}\,\vec{r} = \ddot{\vec{r}}_p$$

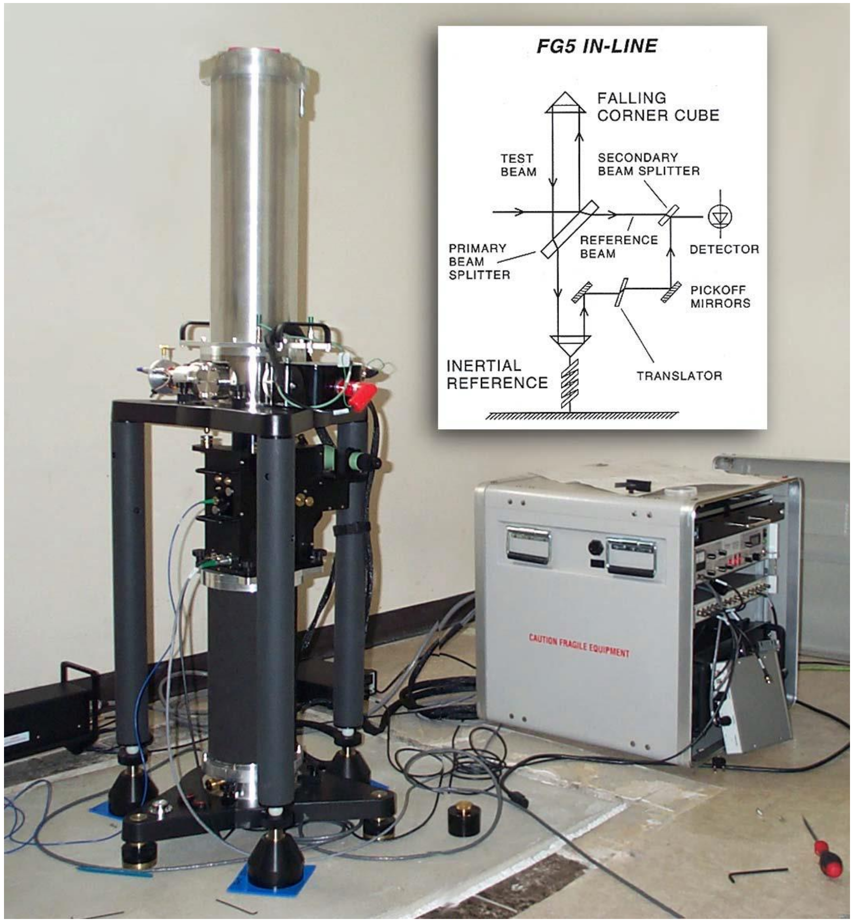
- Perturbations are conservative if only a function of position
  - Satellite does not lose nor gain mechanical energy
  - Exchanges energy between kinetic energy and potential energy
  - Specific mechanical energy is unique for each orbit
- Examples of non-conservative perturbations (changes to $\vec{r}, \vec{v}$)
  - Atmospheric drag
  - Outgassing
  - Tidal effects

Perturbing accelerations – Conservative (2/2)

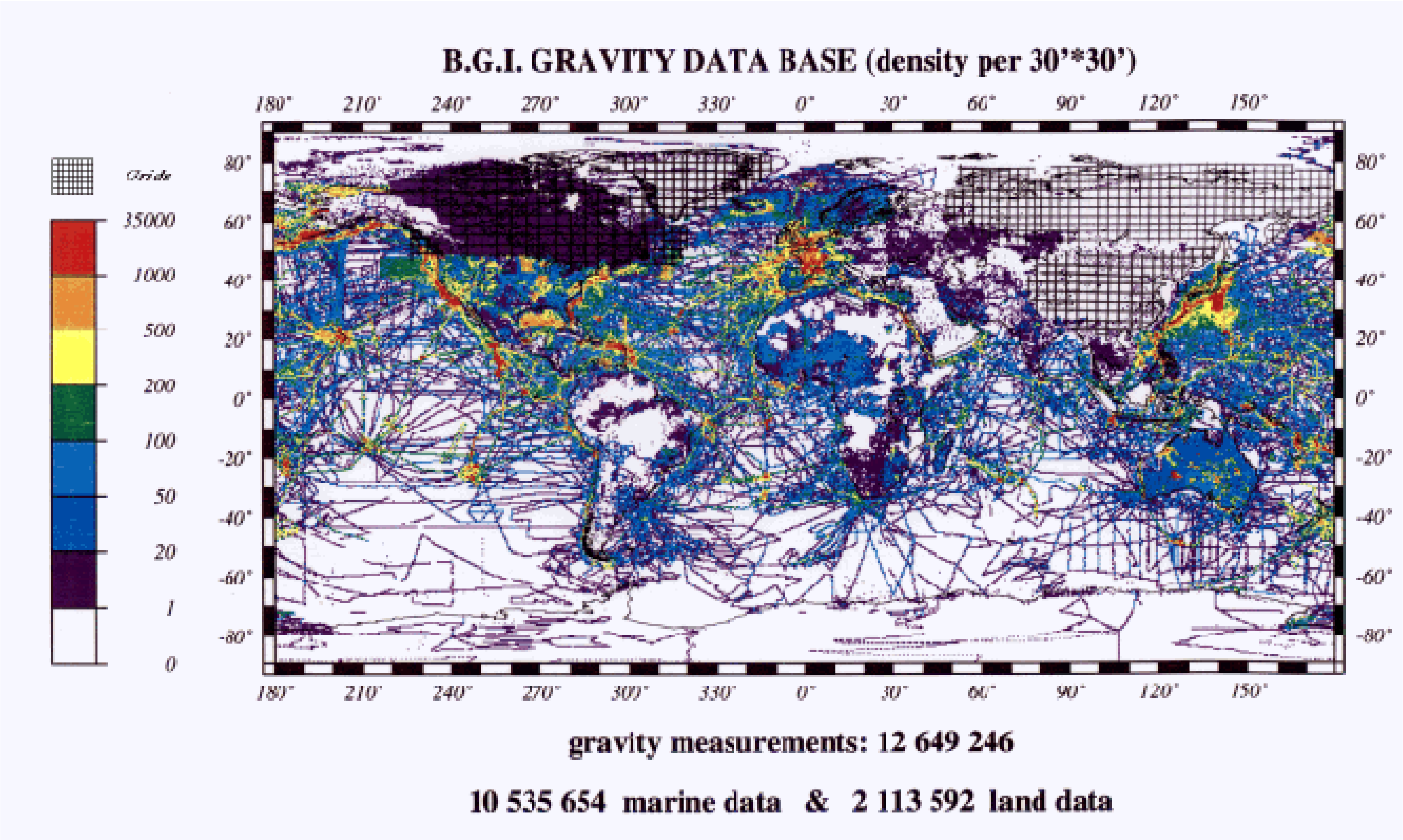
- Examples of conservative perturbations
  - N-body (celestial body) attractions
  - Nonspherical celestial bodies
  - Solar-radiation pressure
- Focus on the gravitational field effects
  - Nonspherical celestial bodies
  - Tidal effects
  - N-body attractions

Perturbing accelerations – Gravitational Models (1/6)

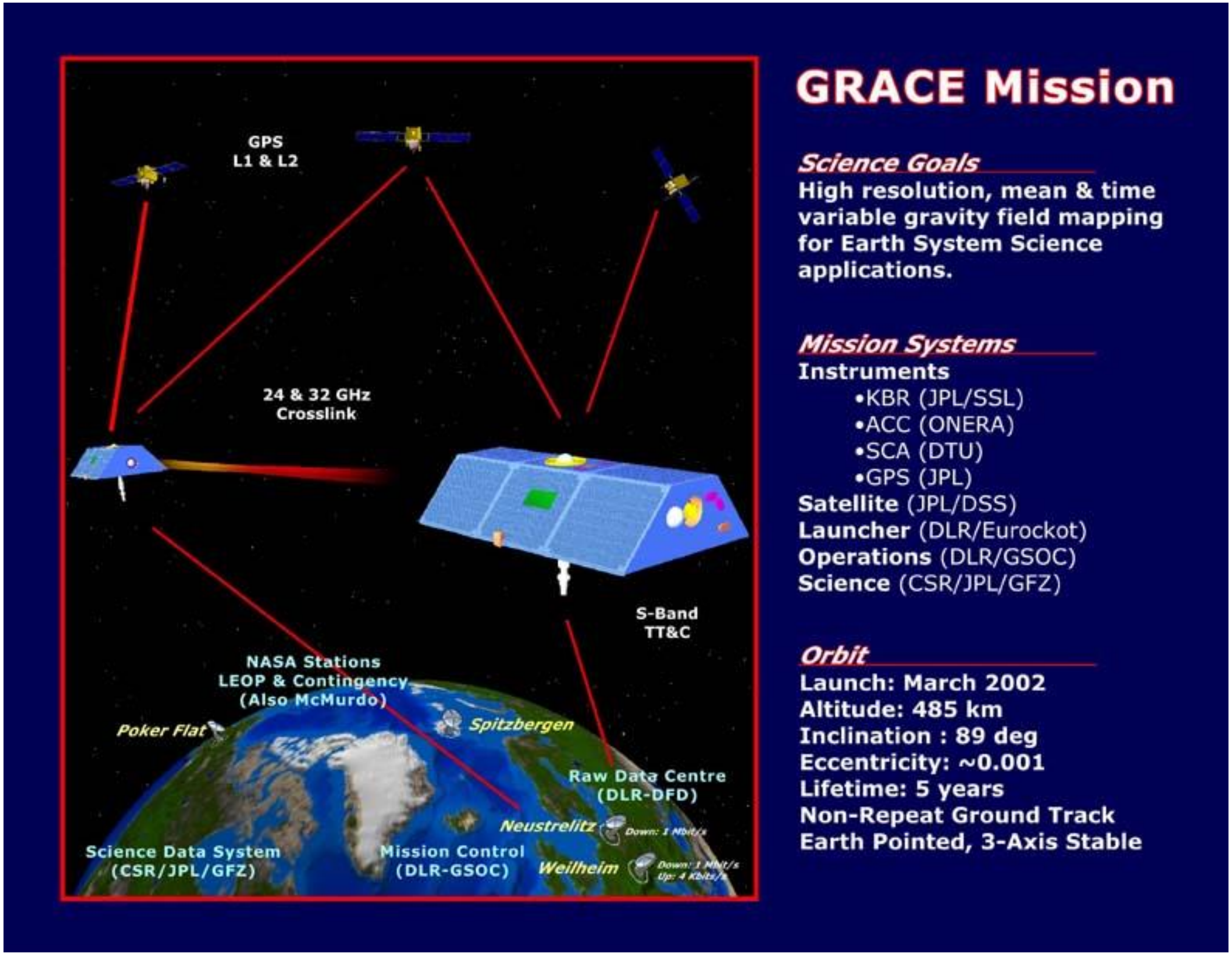
- Terrestrial Measurements



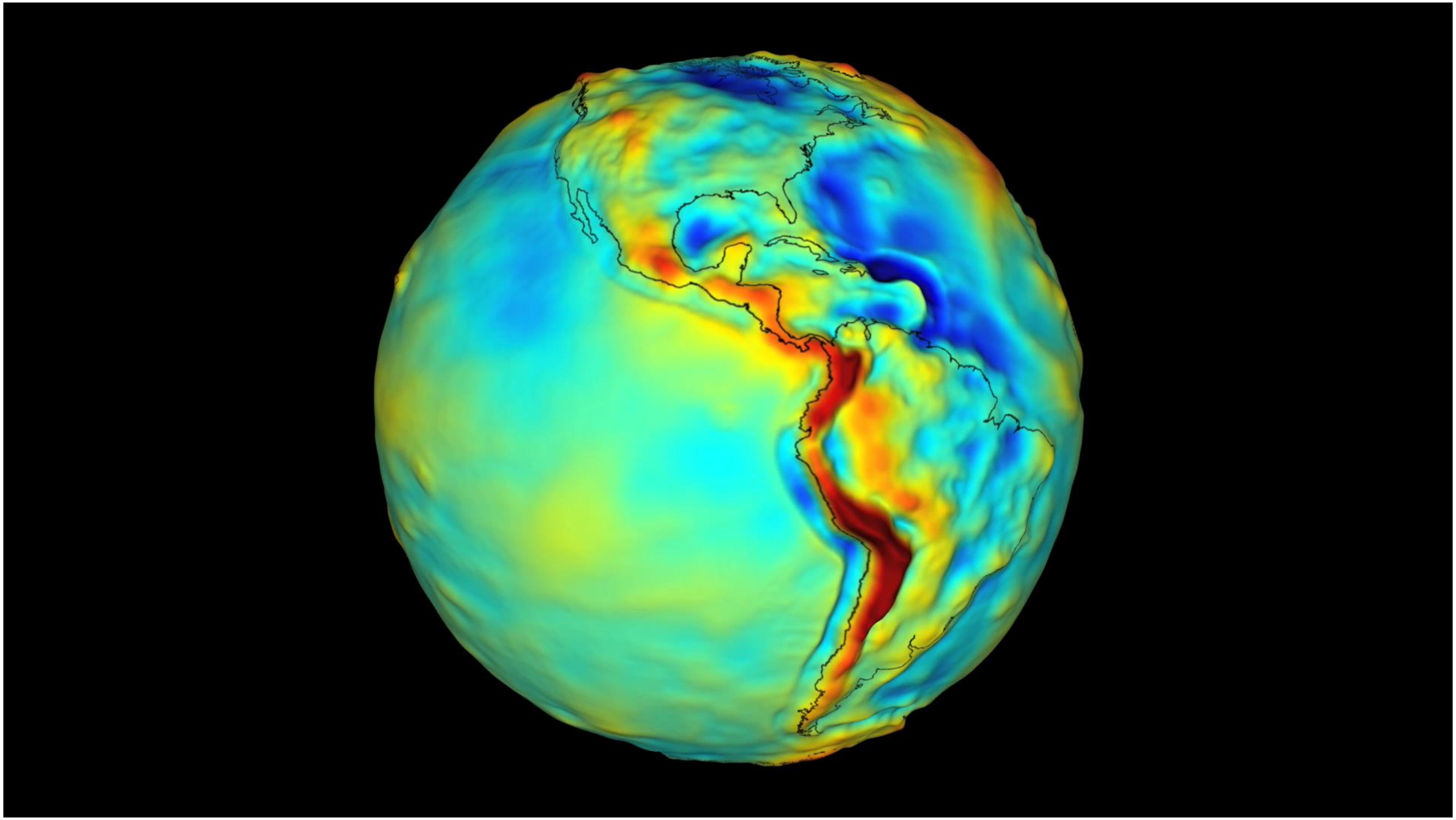
Perturbing accelerations – Gravitational Models (2/6)



Perturbing accelerations – Gravitational Models (3/6)

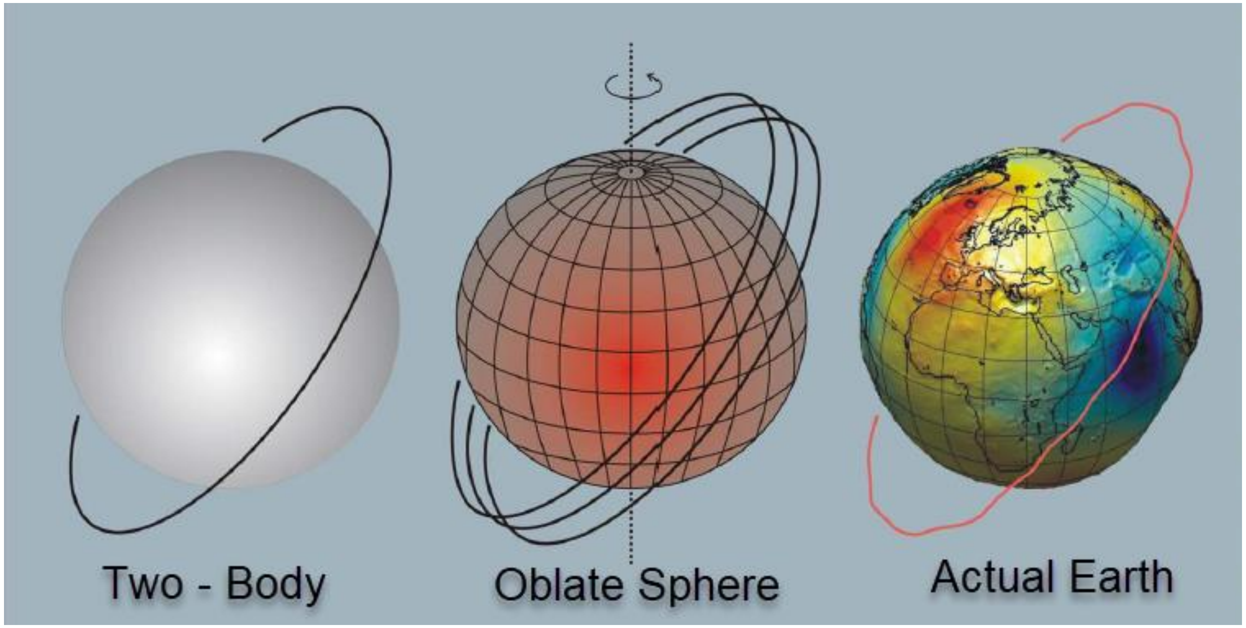


Perturbing accelerations – Gravitational Models (4/6)



Perturbing accelerations – Gravitational Models (5/6)

- Earth's Oblateness ($J_{2,0}$)
  - Bulging at the equator
  - ~400 times larger than the next term
  - When included in satellite orbits maintains reasonable accuracy



Perturbing accelerations – Gravitational Models (6/6)

- Earth's bulge at equator pulls satellite down faster
  - Exerts a force component toward the equator
- Satellite reaches equator short of point for spherical Earth
  - East-bound satellite goes west
  - West-bound satellite goes east

$$\dot{\Omega} = -\frac{9.9358}{(1-e^2)^2}\left(\frac{r_{eq}}{r_{eq}+\bar{h}}\right)^{3.5} \cos i \ \ [\text{deg/mean solar day}]$$

- Secular motion of perigee too
  - Force is no longer proportional to inverse square radius

$$\dot{\omega} = \frac{9.9358}{(1-e^2)^2}\left(\frac{r_{eq}}{r_{eq}+\bar{h}}\right)^{3.5} \left(2 - \frac{5}{2}\sin^2 i\right) \ [\text{deg/mean solar day}]$$