Statistical Orbit Determination

Lecture 4 – Classical Two-Body Problem

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Recap

- Lecture 3 – Notes posted here
  - Problem solution and review of linearization
- Questions
  - Post them to YouTube page

Agenda

- Two-body orbital dynamics
  - General properties
  - Planar motion
  - Kepler’s problem
  - Motion in space
  - Greenwich angle
- Orbit elements to position and velocity
  - State transition matrix and error propagation
- Assigned Problems

Two body – General properties (1/2)

- Newton's Law of Universal Gravitation

$$\vec{F} = -\frac{GmM}{r^2}\frac{\vec{r}}{r}$$

- Assumptions
  - Two point masses/bodies are spherically symmetric
  - Gravitational force propagates instantaneously (No relativistic effects)
- Constants
  - Gravitational constant, $G = 6.6742 \times 10^{-20}\,\frac{\mathrm{km}^3}{\mathrm{kg{-}s}^2}$
  - Earth's estimated mass, $M_\oplus = 5.9722 \times 10^{24}\ \mathrm{kg}$
  - Gravitational parameter, $\mu_\oplus \cong M_\oplus G = 3.98 \times 10^5\,\frac{\mathrm{km}^3}{\mathrm{s}^2}$

Two body – General properties (2/2)

- Equations of motion
  - Equation of relative motion from Laws of Motion and Gravitation

$$\ddot{\bar{r}} + \frac{\mu}{r^3}\bar{r} = 0$$

- Motion of two-body center of mass
  - Center of mass of two spheres moves in a straight line with constant velocity
  - $\bar{R}_{cm} = \frac{M_1\bar{R}_1 + M_2\bar{R}_2}{M_1 + M_2}$, $M_1\ddot{\bar{R}}_1 = \frac{GM_1M_2\bar{r}}{r^3}$, $\rightarrow \ddot{\bar{R}}_{cm} = 0$
- Angular momentum
  - Orbital motion is planar and perpendicular to angular momentum
  - $\bar{r} \times \ddot{\bar{r}} = \bar{r} \times \left(-\frac{\mu}{r^3}\bar{r}\right) \rightarrow \frac{d}{dt}(\bar{r} \times \dot{\bar{r}}) = 0$
- Energy
  - Energy per unit mass is constant
  - $\varepsilon = \frac{\bar{r} \cdot \bar{r}}{2} - \frac{\mu}{r}$

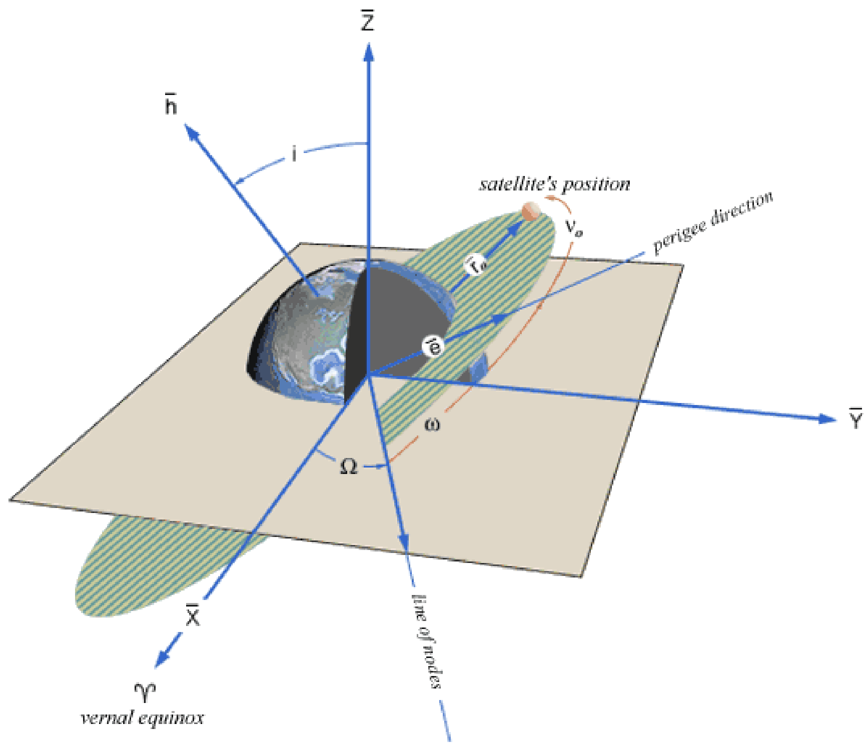
Two body – Planar motion (1/3)

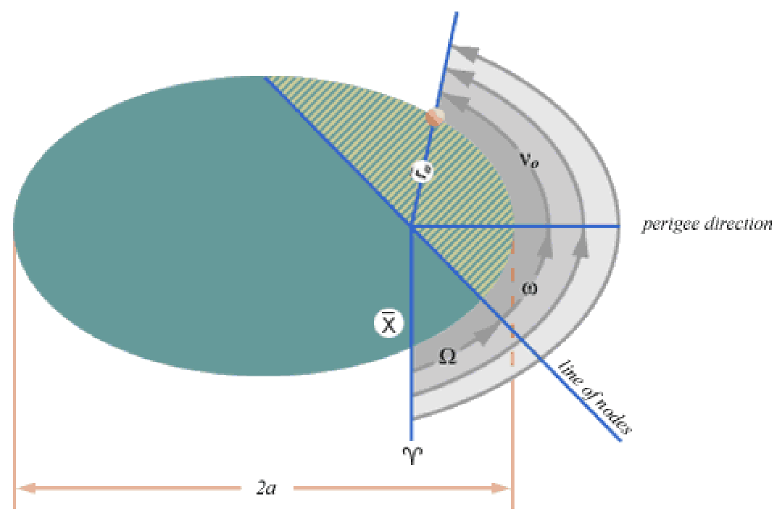
- Orbital motion of M_2 w/r to M_1
  - Takes place in orbital plane
  - Orbital plane is orthogonal to angular momentum, $\bar{h} = \bar{r} \times \dot{\bar{r}}$
- When considered in terms of an inertial nonrotation frame
  - Intersection between (X, Y) plane and orbit plane is line of nodes
  - Ω is angle between X and line of nodes
  - i is inclination or tilt from Z

Two body – Planar motion (2/3)

- Polar coordinates to describe motion in plane
  - $\bar{u}_r$: $\ddot{r} - r\dot{\theta}^2 = -\mu/r^2$
  - $\bar{u}_r$: $2\dot{r}\dot{\theta} + r\ddot{\theta} = 0$
- Using substitution, we can show energy relationship to e
  - $e = \left[1 + \frac{2\varepsilon h^2}{\mu^2}\right]^{\frac{1}{2}}$
- And use the planar motion for
  - $r = \frac{h^2/\mu}{1+e\cos(\theta-\omega)} = \frac{p}{1+e\cos v}$

Two body – Planar motion (3/3)





a - defines the size of the orbit
e - defines the shape of the orbit
i - defines the orientation of the orbit with respect to the Earth's equator.
ω - defines where the low point, perigee, of the orbit is with respect to the Earth's surface.
Ω - defines the location of the ascending and descending orbit locations with respect to the Earth's equatorial plane.
ν - defines where the satellite is within the orbit with respect to perigee.

Two body – Kepler's problem (1/2)

- Unfortunately, no simple relation between v, true anomaly, and t
  - Useful geometrical relationships

$$p = a(1-e^2) \qquad r_p = a(1-e) \qquad r_a = a(1+e)$$

$$b = a\sqrt{1-e^2} \qquad h^2 = \mu p = \mu a(1-e^2)$$

$$n = \sqrt{\mu/a^3} \qquad T = 2\pi\left(\frac{1}{n}\right) \qquad \varepsilon = -\frac{\mu}{2a}$$

  - From the conic section, if v is known then distance, r, can be determined
  - Most often t is known rather than v
- Kepler's equation
  - To transform between time and true anomaly we use an alternate angle
  - E, eccentric anomaly

$$E - e\sin E = n(t - t_p)$$

Two body – Kepler's problem (2/2)

- Kepler's equation

$$E - e\sin E = n\left(t - t_p\right)$$

  - Can now be solved using iterative methods (i.e. Newton Raphson)

$$g = E - e\sin E\ - M$$

  - Where $M = n(t - t_p)$ is the mean anomaly

$$E_{k+1} = E_k - \left(\frac{g}{g'}\right)_k$$

  - $g' = \frac{dg}{dE} = 1\ - e\cos E$ and k is the iteration number

$$r = a(1 - e\cos E)$$

Two body – Motion in Space

- Move to three-dimensional space (out of just the orbit plane)
  - Need 2 additional parameters for orbit plane location in 3D space
    - i, orbital inclination,
    - Ω, right ascension of the ascending node or longitude of ascending node
  - $\cos i = \frac{h_z}{\bar{h}}$ where $0 \leq i \leq 180°$ and $h = \left\|\bar{h}\right\|$
  - $\sin \Omega = \frac{h_X}{h_{XY}}$, $\cos \Omega = -\frac{h_Y}{h_{XY}}$, where $h_{XY} = \sqrt{h_X^2 + h_Y^2}$ and $0 \leq \Omega \leq 360°$

Two body – Greenwich Angle

- Consider the Sun moving relative to the Earth
  - Include a body-fixed coordinate system with (x, y) in equatorial plane
  - x axis is coincident with intersection of Greenwich meridian and equator
  - z axis is coincident with the Earth's angular velocity vector, $\omega_\oplus$
- Special terms
  - Orbit plane of Sun about Earth is the ecliptic
  - Inclination, $i = \epsilon \cong 23.5°$, is the obliquity of the ecliptic
  - Ascending and descending nodes are vernal and autumnal equinoxes, respectively
- Greenwich angle, α_G
  - Defines orientation of Earth-fixed (x, y, z) w/r to vernal equinox
  - $\alpha_G = \omega_\oplus(t - t_o) + \alpha_{G_0}$ where $\omega_\oplus = 2\pi/86164$ rad/sec

Kepler to $\bar{X}$

- Temporal variations exist in location of vernal equinox
  - Necessary to designate a specific epoch for ϵ and ♈
- Assuming initial state is known
  - Already determined i and Ω, see a previous slide
  - a, e, ω, and t_p or M_0 left
  - Using ε, specific energy, the semimajor axis is $a = -\frac{\mu}{2\varepsilon}$
  - $e = \left[1 + \frac{2\varepsilon h^2}{\mu^2}\right]^{\frac{1}{2}}$
  - $\omega = (\omega + v) - v$, where we determine $(\omega + v)$ and v

$$\cos v = \frac{1}{r_0 e}[p - r_0] \qquad \sin v = \frac{p}{he}\frac{\bar{r}_0 \cdot \bar{r}_0}{r_0}$$

$$\cos(\omega + v) = \frac{X_0}{r_0}\cos\Omega + \frac{Y_0}{r_0}\sin\Omega \qquad \sin(\omega + v) = \frac{Z_0}{r_0 \sin i}$$

Kepler to $\bar{X}$

- Eccentric anomaly and mean anomaly are found by

$$\cos E_0 = \frac{r_0}{a}\cos v + e \quad \sin E_0 = \frac{r_0}{b}\sin v$$

$$M_0 = E_0 - e\sin E_0 \qquad t_p = t_0 - \frac{M_0}{n}$$

Kepler to $\bar{X}$ – State Transition Matrix

- State transition matrix, matrizant, propogates X and ϵ
  - $\Delta X(t) = \phi(t, t_0)\Delta \mathrm{X}(\mathrm{t}_0)$ where

$$\phi(t, t_0) = \begin{bmatrix} \frac{\partial X}{\partial X_0} & \cdots & \frac{\partial X}{\partial \dot{Z}_0} \\ \vdots & \ddots & \vdots \\ \frac{\partial \dot{Z}}{\partial X_0} & \cdots & \frac{\partial \dot{Z}}{\partial \dot{Z}_0} \end{bmatrix}$$

Practice problems: The Orbit Problem

PREDICT THE ORBIT

Assigned Problems - Overview

- You are given three problems involving orbital motion. They have been picked to ensure you have a sufficient understanding of orbital mechanics before proceeding. The problems resemble numbers 4, 5, 6, 10, 11, and 12 from the textbook.
- These problems should be complete by Monday, February 4.

Assigned Problems – Problem 1

- Given the following position and velocity of a satellite
  - Expressed in a non-rotating geocentric coordinate system

	Position (m)	Velocity (m/s)
X	7088580.789	-10.20544809
Y	-64.326	-522.85385193
Z	920.514	7482.075141

a) Determine the six orbital elements (a, e, i, Ω, ω, M_0)

b) Assuming X_0 is given and two-body motion, predict position and velocity at $t = 3{,}000$ sec. Determine flight path angle at this time.

c) Determine the latitude and longitude of the subsatellite point for $t = 3{,}000$ sec if α_G at $t = 0$ is 0. Assume the Z axis of the nonrotating system is coincident with the z axis of the rotating system.

Assigned Problems – Problem 2 (1/2)

- Orbit of CRISTA-SPAS-2
  - Cryogenic Infrared Spectrometers and Telescopes for the Atmosphere

The joint venture of DLR and NASA, the small free-flying satellite contains three telescopes, four spectrometers, and a GPS receiver on-board. It is deployed from the shuttle Discovery on STS-85 in August 1997. Using on-board navigation, the receiver measurements are processed in an Earth-centered, Earth-fixed coordinate system.

August 18, 1997

GPS-T (hrs:min:sec)	00:00:0.000000	00:00:03.000000
x	3325396.441	3309747.175
y	5472597.483	5485240.159
z	-2057129.050	-2048664.333

August 19, 1997

GPS-T (hrs:min:sec)	00:00:0.000000	00:00:03.000000
x	4389882.255	4402505.030
y	-4444406.953	-4428002.728
z	-2508462.520	-2515303.456

Assigned Problems – Problem 2 (2/2)

a) Demonstrate that the node location is not fixed in space and determine an approximate rate of node change (degrees/day) from these positions. Compare the node rate with the value predicted by

$$\dot{\Omega} = -\frac{3}{2} J_2 \, \frac{n}{(1-e^2)^2} \left(\frac{a_e}{a}\right)^2 \cos i$$

b) Determine the inclination of CRISTA-SPAS-2 during the first 3-sec interval and the last 3-sec interval.

Comment: The position vectors determined by GPS in this case are influenced at the 100-meter level by Selective Ability, but the error does not significantly affect this problem.

Assigned Problems – Problem 3 (1/2)

- GLONASS
  - Russia's answer for American GPS

Given a set of initial conditions for a high-altitude GLONASS satellite, numerically integrate the equations of motion for one day.

a) Assuming the satellite is influenced by J_2 only, derive the equations of motion in non-rotation coordinates. Assume the nonrotating Z axis coincides with the Earth-fixed z axis.

b) During the integration, compute the Jacobi constant and the Z component of the angular momentum. Are these quantities constant?

c) Plot the six orbital elements as a function of time.

d) Identify features similar to and different from Fig. 2.3.5

Assigned Problems – Problem 3 (2/2)

e) Compare the node rate predicted by

$$\dot{\Omega} = -\frac{3}{2} J_2 \frac{n}{(1-e^2)^2} \left(\frac{a_e}{a}\right)^2 \cos i$$

with a value estimated from (c).

f) Compare the amplitude of the semimajor axis periodic term with

$$a(t) = \bar{a} + 3\bar{n}\bar{a}J_2 \left(\frac{a_e}{\bar{a}}\right)^2 \sin^2 \frac{\bar{\imath}(\cos(2\omega + 2M))}{2\dot{\omega}_s + 2\dot{M}_s}$$

g) Plot the ground track. Does the ground track repeat after one day?

a	e	i	Ω	ω	M_0
25500.0 km	0.0015	63 deg	-60 deg	0 deg	0 deg