Simpson Aerospace: Solution to Homework 1

Objective: Explain how the Newton-Raphson method is used for vectors

TO: Statistical Orbit Determination Class
CC:
REF:

Date: 9 Jul 2018
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SUMMARY:

Problem 1 provides the solution for us. We are learning how to use iterative methods to estimate the state vector. In this case we will use the Newton-Raphson root-finding method to solve for the problem. An Excel sheet is provided that walks through the first three iterations as an illustration. C++ code is provided on GitHub that will solve for the final solution and show the number of iterations.

OVERVIEW:

Given a set of unitless initial conditions, we are told that only the range observations and the ground station position are accurate. We will solve iteratively using the Newton iteration scheme covered previously.

$$X_0 = \begin{bmatrix} x_0 = 1.5 \\ y_0 = 10.0 \\ \dot{x}_0 = 2.2 \\ \dot{y}_0 = 0.5 \\ g = 0.3 \\ x_s = 1.0 \\ y_s = 1.0 \end{bmatrix}$$

Time	Range Observation, ρ
0	7.0
1	8.00390597
2	8.94427191
3	9.801147892
4	10.630145813

We are given the final answer to confirm our iterative solution.

Solution

$$x_0 = 1.0$$
$$y_0 = 8.0$$
$$\dot{x}_0 = 2.0$$
$$\dot{y}_0 = 1.0$$
$$g = 0.5$$

Solution

The pseudo-range, $\rho(x, y)$, for the uniform gravitational field is determined by $\rho = \sqrt{(x - x_s)^2 + (y - y_s)^2}$. x and y are both dependent on t. y is also dependent upon g in this two-dimensional field. This is important because in order to use the Newton iteration method, we need to find $\dot{\rho}$. We do this by taking a first-order Taylor series expansion and neglecting higher terms.

$$\rho(x + \delta x, y + \delta y, \dot{x} + \delta\dot{x}, \dot{y} + \delta\dot{y}, g + \delta g) = \rho(x, y, \dot{x}, \dot{y}, g) + \frac{\partial\rho}{\partial x}\delta x + \frac{\partial\rho}{\partial y}\delta y + \frac{\partial\rho}{\partial\dot{x}}\delta\dot{x} + \frac{\partial\rho}{\partial\dot{y}}\delta\dot{y} + \frac{\partial\rho}{\partial g}\delta g$$

The term, $\rho(x + \delta x, y + \delta y, \dot{x} + \delta\dot{x}, \dot{y} + \delta\dot{y}, g + \delta g)$ is the next iteration of the iteration scheme. Subtracting the next iteration from the current provides us with the residuals or $\delta\rho$. Multiplying these by the inverse of the mapping matrix gives us the residuals of the state vector.

$$\delta\rho = \rho(x + \delta x, y + \delta y, \dot{x} + \delta\dot{x}, \dot{y} + \delta\dot{y}, g + \delta g) - \rho(x, y, \dot{x}, \dot{y}, g)$$

$$\delta\rho = \frac{\partial\rho}{\partial x}\delta x + \frac{\partial\rho}{\partial y}\delta y + \frac{\partial\rho}{\partial \dot{x}}\delta\dot{x} + \frac{\partial\rho}{\partial \dot{y}}\delta\dot{y} + \frac{\partial\rho}{\partial g}\delta g$$

$$\delta\rho = [H][\delta u]$$

$$H = \begin{bmatrix} \frac{\partial\rho_1}{\partial x} & \cdots & \frac{\partial\rho_1}{\partial g} \\ \vdots & \ddots & \vdots \\ \frac{\partial\rho_n}{\partial x} & \cdots & \frac{\partial\rho_n}{\partial g} \end{bmatrix}$$

$$\delta u = \begin{bmatrix} \delta x_0 \\ \delta y_0 \\ \delta\dot{x}_0 \\ \delta\dot{y}_0 \\ \delta g \end{bmatrix}$$

$$\delta u = H^{-1}\delta\rho$$

Having found the residuals of our state vector we can now add this to our original state vector X_0 to determine the next iteration of the state vector. The Excel spreadsheet shows this process very clearly without calculating the relative error for each step.