Statistical Orbit Determination

Lecture 5 – Perturbed Motion

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Recap

- Lecture 4 – Notes posted here
  - Classical two-body problem
- Lecture 6 – Coordinate systems and time
- Questions
  - Post them to YouTube page

Agenda

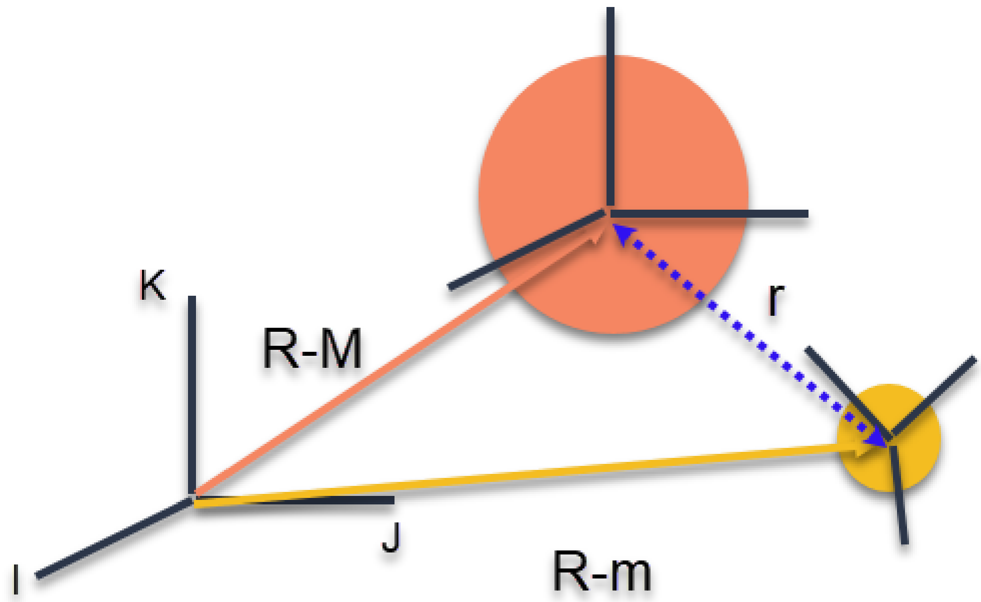
- Perturbed Motion
  - Classical example: Lunar problem
  - Variation of parameters
  - Gravitational perturbations
    - Oblateness
    - Third-body Effects
  - Nongravitational perturbations
- Assigned Problems

Perturbed Motion – Recap of two body (1/2)

- Relative motion of m_2 w/r to m_1

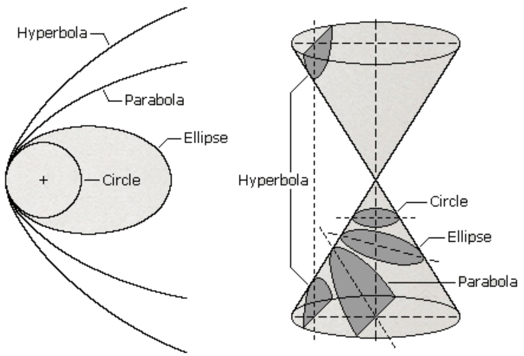
$$\ddot{\vec{r}} + \frac{\mu}{r^3}\,\vec{r} = 0$$

- Assumptions
  - Two point masses/bodies are spherically symmetric
  - Gravitational force propagates instantaneously (No relativistic effects)



Perturbed Motion – Recap of two body (2/2)

- Previous description of motion is idealized
  - Equations of motion can be solved analytically
  - Motion simplified to a geometric shape (circle, ellipse, parabola, hyperbola)
- Newton
  - Told Halley, that the motion of the Moon [in the three body system] “made his head ache and kept him awake so often that he would think of it no more,” (Moulton, p. 363, 1914).
- No general closed-form solution for three-body problem
  - Approximate analytical solutions use two-bodies as a reference
  - Approximate or general perturbations solution adds contributing perturbing forces
- Numerical solutions
  - Perturbed motion represented by a set of ODEs with specified initial conditions (special perturbations)

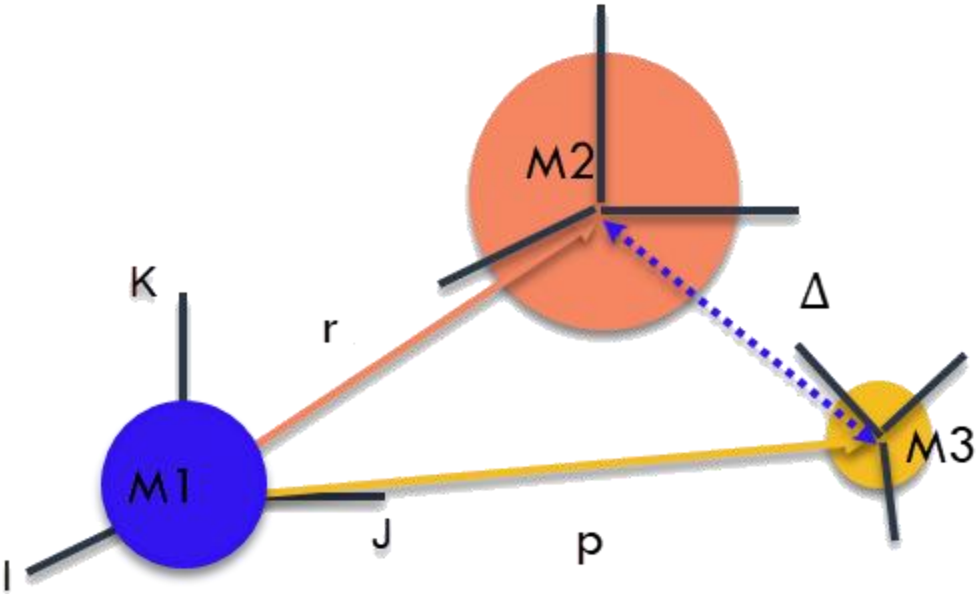


Perturbed Motion – Lunar problem (1/2)

- Derive equations of motion for three body problem
  - Solve EoM using numerical integration

$$\ddot{\bar{r}} = -\frac{\mu\bar{r}}{r^3} + GM_3\left(\frac{\bar{\Delta}}{\Delta^3} - \frac{\bar{r}_p}{r_p^3}\right)$$
$$\ddot{\bar{r}}_p = -\frac{\mu'\bar{r}}{r^3} + GM_2\left(\frac{\bar{\Delta}}{\Delta^3} + \frac{\bar{r}}{r^3}\right)$$

  - where $\mu = G(M_1 + M_2)$ and $\mu' = G(M_1 + M_3)$
- Let M_1, M_2, and M_3 represent Earth, the Moon, and the Sun, respectively



Perturbed Motion – Lunar problem (2/2)

- Restricted three-body problem (Szebehely, 1967)
  - Sun's mass is more than 300,000 times greater than Earth
  - Moon's mass is 81 times smaller than Earth

$$\mu = GM_1$$
$$\mu' = GM_3$$

- Can integrate w/o approximation given specified initial conditions
  - (Shampine and Gordon, 1975), since Sun is dominant perturbation of the lunar motion
  - Rotate about X in ECI frame so Z axis is perpendicular to ecliptic
  - Can show osculating elements (orbital elements) are not constant due to perturbations
- Osculating element variation
  - Ascending node linear variation with time (secular variation/periodic variations)
  - Inclination has no apparent secular but experiences periodic variation
  - Secular node rate is negative (regression of the node, $\sim 19.4°$ per year)

Perturbed Motion – Variation of parameters

- Temporal variations of Moon's osculating elements

$$\ddot{\vec{r}} = -\frac{\mu}{r^3}\,\vec{r} + \bar{f}$$

  - Develop solution to ODEs by using variation of parameters
  - $\bar{f}$ is perturbing force
  - See Appendix D for differential equations describing change of osculating elements
- In some cases $\bar{f}$ is derivable from potential or disturbing function
- $\bar{f}$ can be categorized as gravitational or nongravitational

Gravitational – Mass Distribution (1/6)

- Mass distribution
  - Two point masses gravitational potential

$$U = \frac{GM_1M_2}{r}$$

$$\bar{F} = \nabla U = \frac{GM_1M_2}{r^3}\bar{r}$$

- Can model mass distribution as collection of point masses
  - Potential experienced by point mass, m', is

$$U = m' \int \int \int G\gamma \, dx \, dy \, dz/\rho$$

where γ is the mass density associated with dm, $dx \, dy \, dz$ are differential volume, and ρ is distance between differential mass and external mass m'

- Spherical harmonics splits Earth into regions
  - Allows us to assign mass coefficients/properties to each region

Gravitational – Mass Distribution (2/6)

- Taking the external mass to be unity, $m' = 1$
  - $U = \int_M \frac{Gdm}{\rho}$, where we are integrating over entire mass
  - Position vector of m' is $\bar{r}$
  - Where (x, y, z) is considered to be body-fixed
  - (x, y) equatorial plane and x to Greenwich meridian
- Expand using infinite series
  - Expand $U = \int_M \frac{Gdm}{\rho}$ using infinite series and Legendre polynomials

$$U = \frac{G}{r} \int_M \sum_{l=0}^{\infty} \left(\frac{R}{r}\right)^l P_l(\cos S) dm$$

  - Where R is distance between origin and dm and P_l is Legendre polynomial of degree l with an argument equal to the cosine of the angle between the two vectors $R$and r

Gravitational – Mass Distribution (3/6)

- Expand the Legendre polynomial into spherical harmonics
  - Terms dependent on mass distribution are collected into coefficients

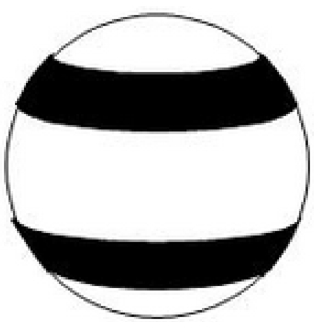
$$U = \frac{\mu}{r} + U'$$

$$U' = -\frac{\mu^*}{r}\sum_{l=1}^{\infty}\left(\frac{a_e}{r}\right)^l P_l(\sin\phi)J_l + \frac{\mu^*}{r}\sum_{l=1}^{\infty}\sum_{m=1}^{l}\left(\frac{a_e}{r}\right)^l P_{lm}(\sin\phi)[C_{lm}\cos m\lambda + S_{lm}\sin m\lambda]$$

- Coordinates of m' are now expressed in spherical coordinates (r, ϕ, λ)
  - ϕ is geocentric latitude
  - λ is longitude angle
- Scale factors to nondimensionalize C_{lm} and S_{lm}
  - Reference mass, $\mu^* = GM^*$
  - Reference distance, a_e

Gravitational – Mass Distribution (4/6)

- Zonal harmonics $(m = 0)$, J_l
  - No dependence on longitude
  - Circle of latitude alternately positive and negative
- Sectorial harmonics $(n = m)$
  - Sectors alternately positive and negative along lines of longitude
- Tesseral harmonics $(n \neq m)$
  - Checkerboard array of domains, "square" harmonics



Zonal Harmonics

Sectorial Harmonics

Tesseral Harmonics

Gravitational – Mass Distribution (5/6)

- Normalized expressions normally used
  - Legendre functions normally high numerical values compared to mass coefficients
- Degree 1 terms, l, proportional to distance between cm and O
- Degree 2 terms proportional to moments and products of inertia

$$\bar{F}^* = m'\nabla\mathrm{U}$$

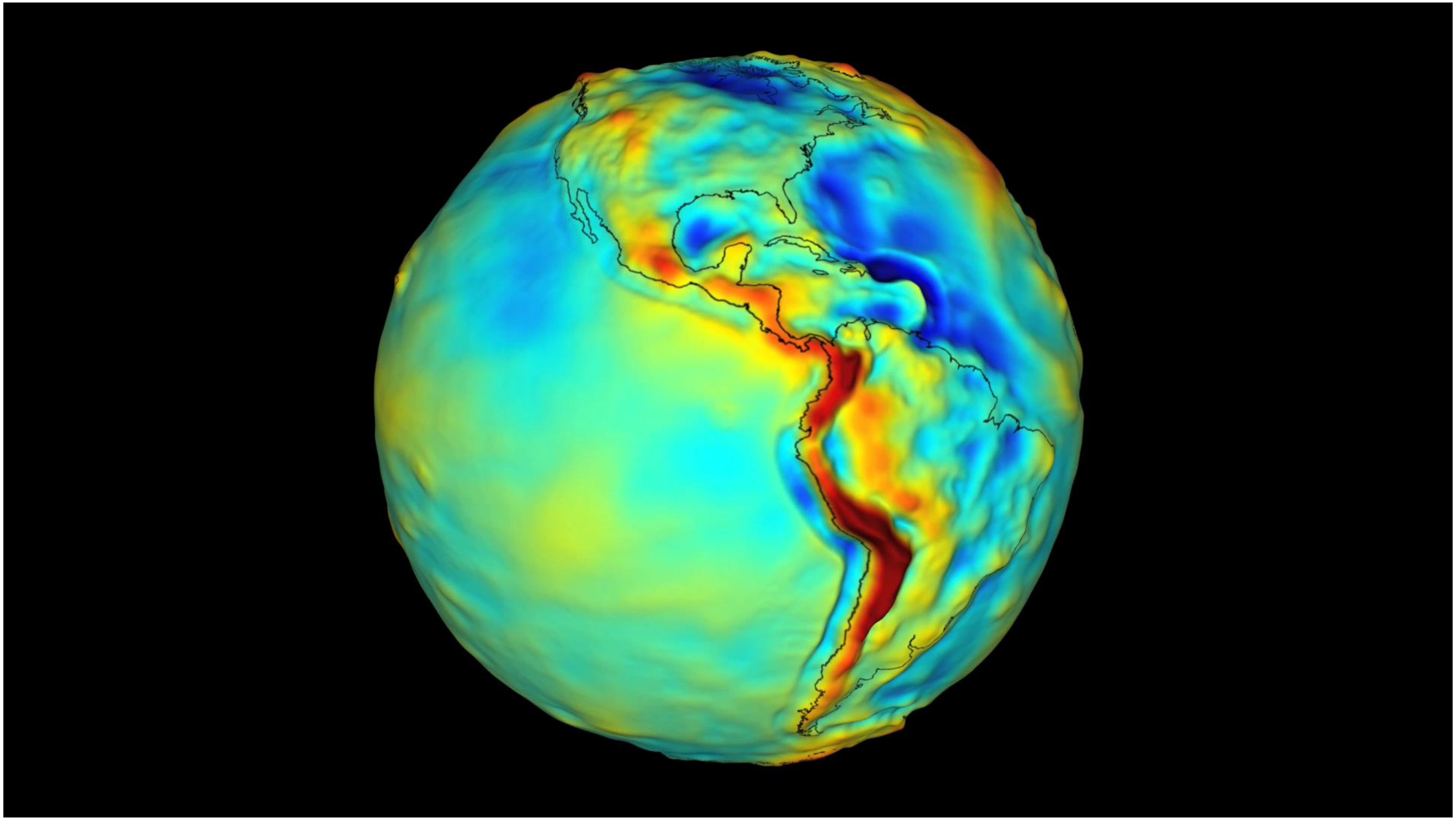
$$\ddot{\bar{r}} = \left(1 + \frac{m'}{M}\right)\nabla\mathrm{U}$$

  - If m'/M is very small

$$\ddot{\bar{r}} = \nabla\mathrm{U} = -\frac{\mu\bar{r}}{r^3} + \bar{f}_{NS}$$

- Be careful about whether system used is nonrotating or rotating

Gravitational – Mass Distribution (6/6)



Gravitational – Oblateness (1/3)

- More than 95% of gravitational force (that's not μ/r^2) is J_2
  - Potential for the ellipsoid of revolution

$$U' = -\frac{\mu}{r}\left(\frac{a_e}{r}\right)^2 J_2 P_2(\sin\phi)$$

- Can relate to orbit elements

$$\sin\phi = \sin i \sin(\omega + v)$$

  - Gravitational potential now can be expressed in terms of orbit elements
  - Use eccentricity expansions

$$U' = -\frac{\mu}{a}\left(\frac{a_e}{a}\right)^2 J_2\{3/4 \sin^2 i[1 - \cos(2\omega + 2M)] - 1/2\} + \text{ higher order terms}$$

  - Can divide contributions into secular and periodic

$$U' = U_s + U_p$$

  - Secular

$$U_s = -\frac{GM}{a}\left(\frac{a_e}{a}\right)^2 J_2\left(\frac{3}{4}\sin^2 i - \frac{1}{2}\right)$$

  - Periodic

$$U_p = \frac{GM}{a}\left(\frac{a_e}{a}\right)^2 J_2\left(\frac{3}{4}\sin^2 i \cos(2\omega + 2M)\right)$$

Gravitational – Oblateness (2/3)

- Which orbit elements affected over time
  - a, e, i not affected by time
  - $\dot{\Omega}_s$, secular node rate is constant for given a, e, i

$$\dot{\Omega}_s \cong -\frac{3}{2} J_2 n \left(\frac{a_e}{a}\right)^2 \cos i$$

- Application in solar-synchronous satellites
  - Constantly aligned with Earth-Sun direction
  - $\dot{\Omega}_s = 360°/365.25$ days $\cong 1°/$day
  - In other words, for $e \approx 0, a \approx 7000$ km, and $i \approx 98$
- Secular rates of Ω, ω, and M (Kaula, 1966)

$$\dot{\Omega}_s = -\frac{3}{2} J_2 \frac{n}{(1-e^2)^2} \left(\frac{a_e}{a}\right)^2 \cos i$$

$$\dot{\omega}_s = \frac{3}{4} J_2 \frac{n}{(1-e^2)^2} \left(\frac{a_e}{a}\right)^2 (5\cos^2 i - 1)$$

$$\dot{M}_s = \bar{n} + \frac{3}{4} J_2 \frac{n}{(1-e^2)^{3/2}} \left(\frac{a_e}{a}\right)^2 (3\cos^2 i - 1)$$

Gravitational – Oblateness (3/3)

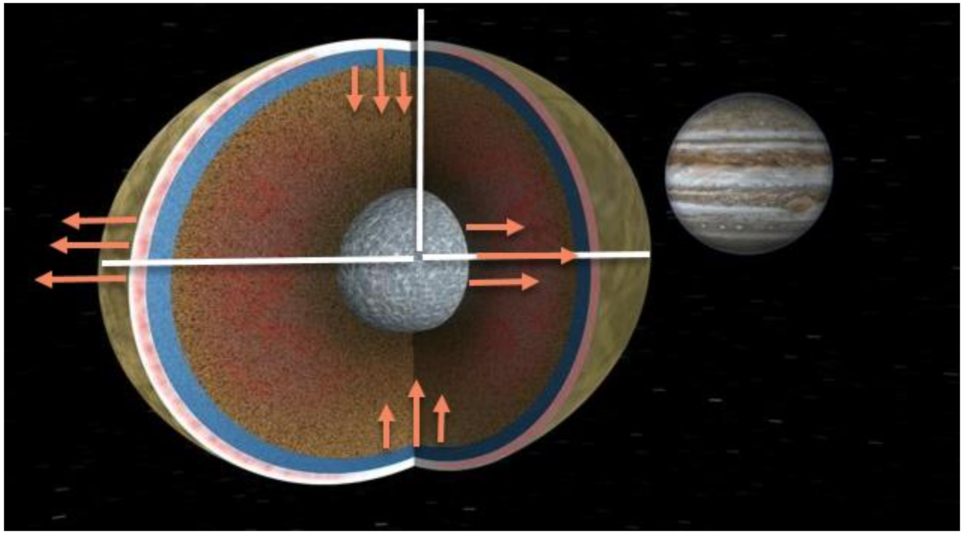
- Previous equations (Kaula, 1966) use mean elements
  - a, e, i have periodic variations averaged out
  - $\bar{n} = \sqrt{\mu/\bar{a}^3}$ where $\bar{a}$ is the mean value
- Can use this for simple model, Secularly Precessing Ellipse
- From Kaula's linear derivation
  - Nodal period, $P_n = \frac{2\pi}{\dot{\omega}_s + \dot{M}_s}$
  - Nodal day, $D_n = \frac{2\Pi}{\dot{\Omega}_s + \omega_e}$

Gravitational – Third-body effects

- Consider the two-body case, then add another body
  - Not considering the non-uniformity of the gravitational field just yet
  - Addition given by $f_{3b} = \sum_{j=1}^{n_p} \mu_j \left(\frac{\bar{\Delta}_j}{\Delta_j^3} - \frac{\bar{\mathrm{r}}_j}{\mathrm{r}_j^3} \right)$
  - $\bar{\Delta}_j$ is the vector between the j body and the satellite
  - $\bar{r}_j$ is the vector between the j body and the Earth

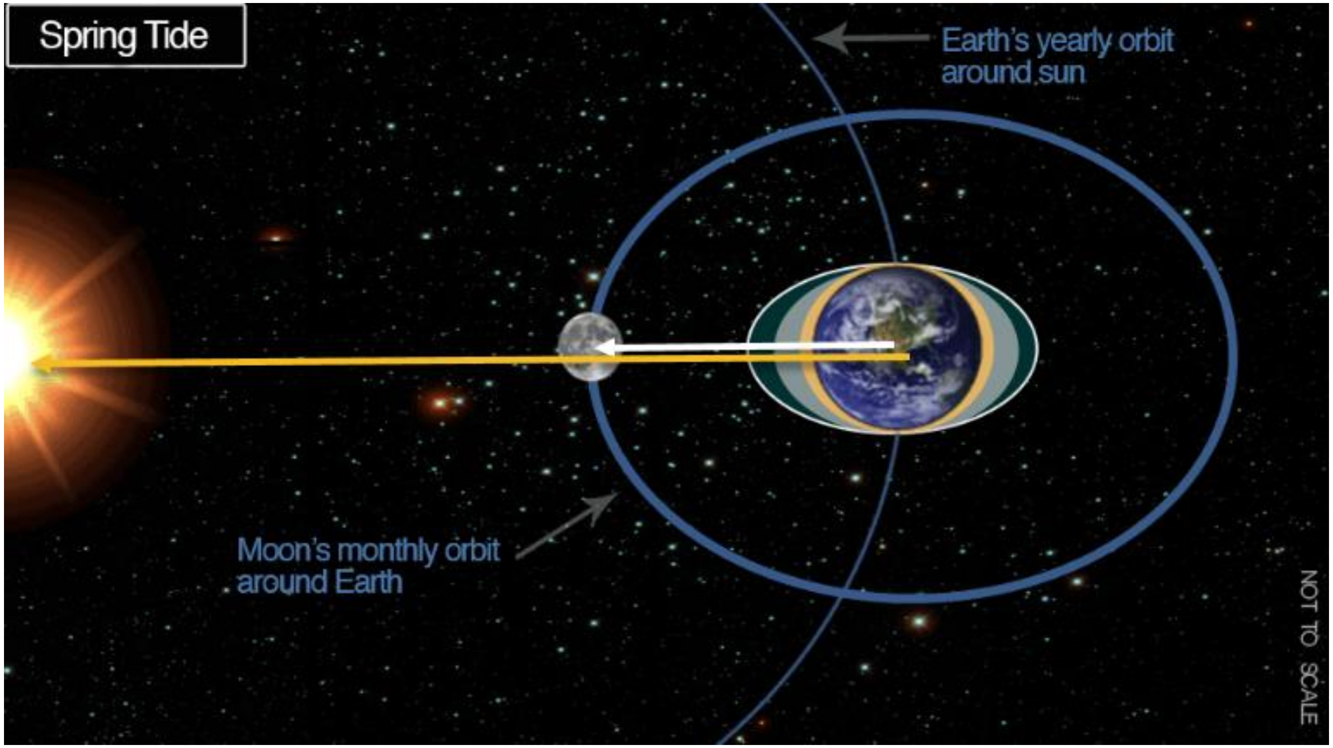
Gravitational – Tidal Effects (1/3)

- One side of celestial body experiences greater acceleration
  - Redistributes liquid and solid mass of body
  - Measured w/r to center of mass
- Rotation of body creates a phase advance
  - Accelerates moon
  - Tidal locking, moon's tidal bulge and face match Earth



Gravitational – Tidal Effects (2/3)

- Changes GS position and SC acceleration
- Sun and Moon tidal interaction
  - Spring Tides and Neap Tides



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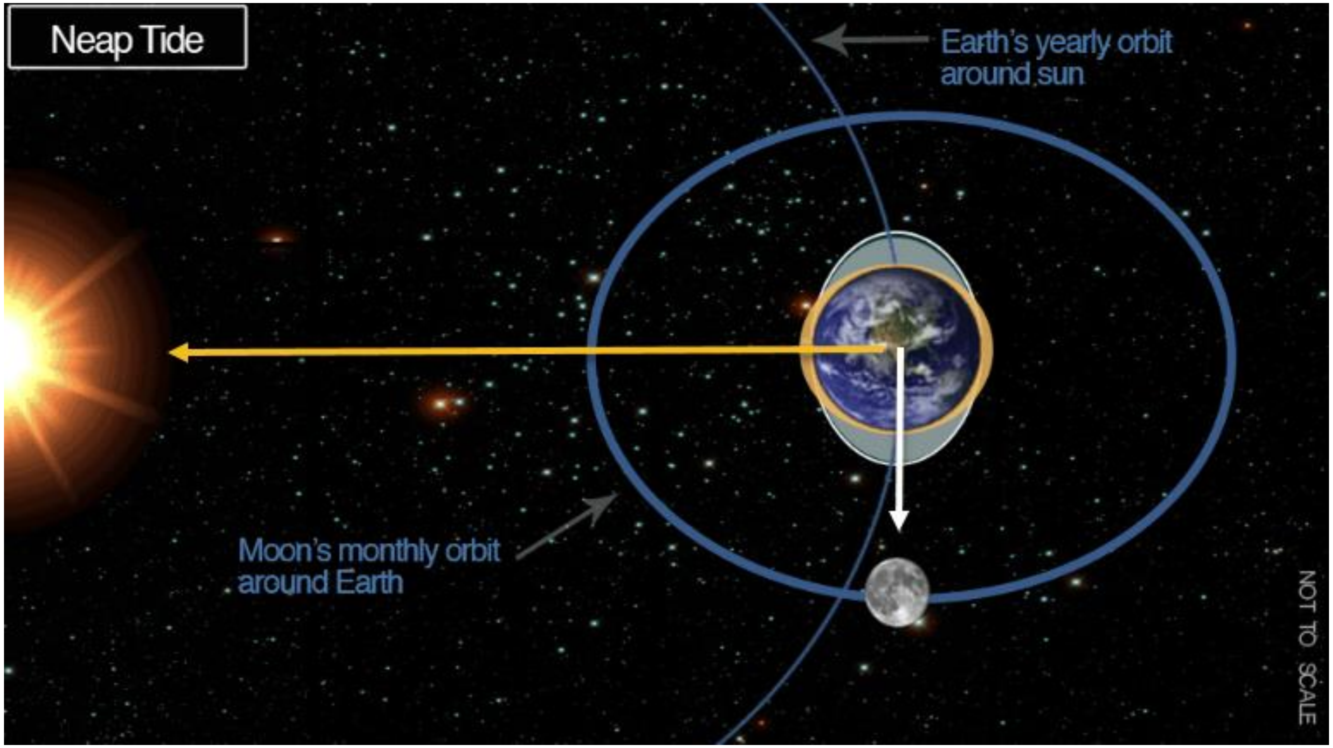


Gravitational – Tidal Effects (3/3)

- Tidal potential approximated by

$$U_t = -\frac{\mu_{\text{moon}}}{R}\sum_{n=2}^{\infty} k_n \left(\frac{r}{R}\right)^n P_n(\cos\psi)$$

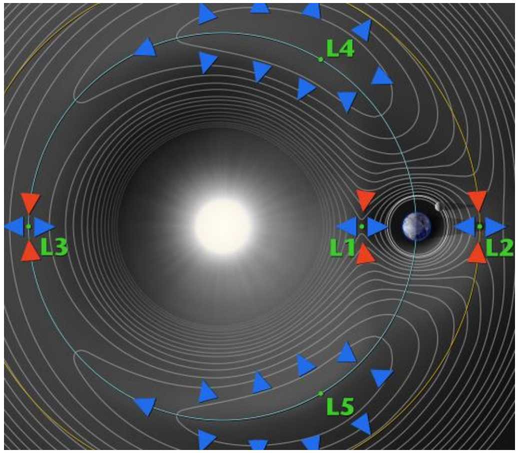
- Love numbers, k_n, describe how deformable celestial body is
  - r, ψ describe position on affected body looking down on orbital plane
- Variations in topology change amplitudes of tides

Celestial Body	Ocean	Solid
Sun, ☉	0.82 ft	0.60 ft
Moon, ☽	1.8 ft	1.3 ft

Applications – Lagrange Points (1/2)

Applications – Lagrange Points (2/2)

- Points where gravitational pull is balanced
  - Exists between all multibody systems
  - Sun-Earth Lagrange points most commonly used/considered
- Several current missions either use or will pass L points
  - OSIRIS-REx will swing by L4 looking for Earth Trojans
  - JWST will orbit L2 to examine early origins of universe
  - Solar and Heliospheric Observatory (SOHO) orbits L1



Perturbed Motion – Nongravitational

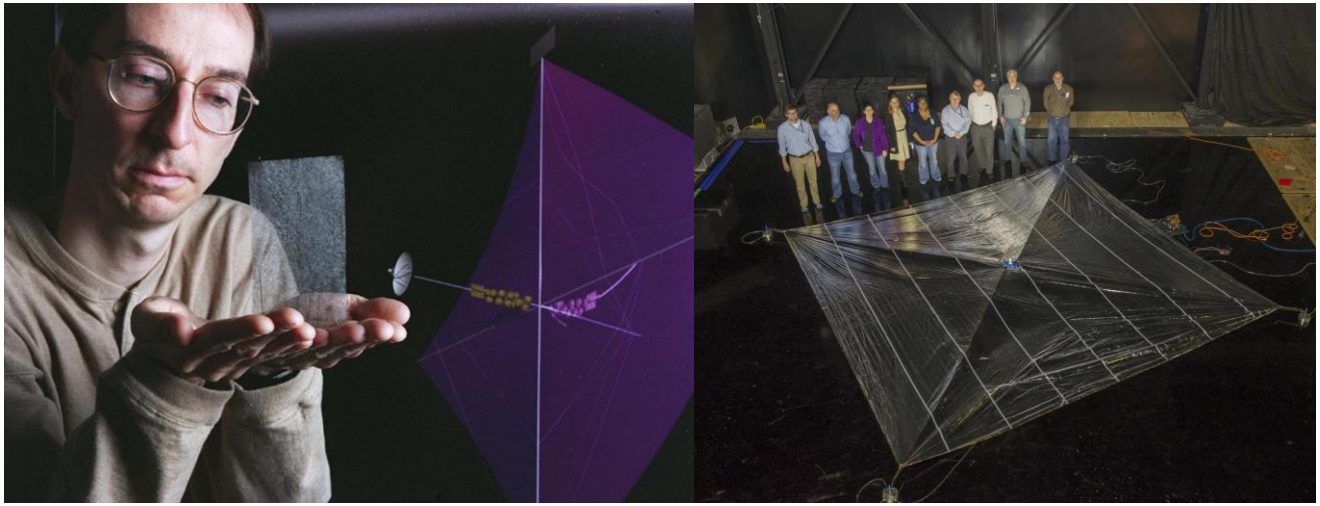
- Atmospheric Resistance (Drag)
  - $\bar{f}_D = -\frac{1}{2}\rho\left(\frac{C_D A}{m}\right) v\bar{v}$, force of drag acting opposite to movement
  - Ballistic coefficient, $\left(\frac{C_D A}{m}\right)$
    - At low altitudes (~ 350 km) the atmospheric density is 10^{-11} of sea level
    - Mean free path increases to meters
  - Drag removes energy from orbit
    - Secular decay in semimajor axis and eccentricity
    - Decay in a will determine lifetime of satellite
- [Some re-entry numbers]
  - $\sim$170 million pieces of space debris
  - Less than 0.2% tracked

Perturbed Motion – Nongravitational

- Solar radiation pressure
  - Transfer of momentum through photons

$$f_{SRP} = -P\frac{vA}{m}C_R\bar{u}$$

    - P is the momentum flux
    - A is cross-sectional area
    - C_R, reflectivity coefficient
    - v, eclipse factor for when satellite is in shadow



Practice problems: The Orbit Problem

PREDICT THE ORBIT

Assigned Problems - Overview

- You are given three problems involving orbital motion. They have been picked to ensure you have a sufficient understanding of orbital mechanics before proceeding. The problems resemble numbers 4, 5, 6, 10, 11, and 12 from the textbook.
- These problems should be complete by Friday, February 8.

Assigned Problems – Problem 1

- Given the following position and velocity of a satellite
  - Expressed in a non-rotating geocentric coordinate system

	Position (m)	Velocity (m/s)
X	7088580.789	-10.20544809
Y	-64.326	-522.85385193
Z	920.514	7482.075141

a) Determine the six orbital elements (a, e, i, Ω, ω, M_0)

b) Assuming X_0 is given and two-body motion, predict position and velocity at $t = 3{,}000$ sec. Determine flight path angle at this time.

c) Determine the latitude and longitude of the subsatellite point for $t = 3{,}000$ sec if α_G at $t = 0$ is 0. Assume the Z axis of the nonrotating system is coincident with the z axis of the rotating system.

Assigned Problems – Problem 2 (1/2)

- Orbit of CRISTA-SPAS-2
  - Cryogenic Infrared Spectrometers and Telescopes for the Atmosphere

The joint venture of DLR and NASA, the small free-flying satellite contains three telescopes, four spectrometers, and a GPS receiver on-board. It is deployed from the shuttle Discovery on STS-85 in August 1997. Using on-board navigation, the receiver measurements are processed in an Earth-centered, Earth-fixed coordinate system.

August 18, 1997

GPS-T (hrs:min:sec)	00:00:0.000000	00:00:03.000000
x	3325396.441	3309747.175
y	5472597.483	5485240.159
z	-2057129.050	-2048664.333

August 19, 1997

GPS-T (hrs:min:sec)	00:00:0.000000	00:00:03.000000
x	4389882.255	4402505.030
y	-4444406.953	-4428002.728
z	-2508462.520	-2515303.456

Assigned Problems – Problem 2 (2/2)

a) Demonstrate that the node location is not fixed in space and determine an approximate rate of node change (degrees/day) from these positions. Compare the node rate with the value predicted by

$$\dot{\Omega} = -\frac{3}{2} J_2 \frac{n}{(1-e^2)^2} \left(\frac{a_e}{a}\right)^2 \cos i$$

b) Determine the inclination of CRISTA-SPAS-2 during the first 3-sec interval and the last 3-sec interval.

Comment: The position vectors determined by GPS in this case are influenced at the 100-meter level by Selective Ability, but the error does not significantly affect this problem.

Assigned Problems – Problem 3 (1/2)

- GLONASS
  - Russia's answer for American GPS

Given a set of initial conditions for a high-altitude GLONASS satellite, numerically integrate the equations of motion for one day.

a) Assuming the satellite is influenced by J_2 only, derive the equations of motion in non-rotation coordinates. Assume the nonrotating Z axis coincides with the Earth-fixed z axis.

b) During the integration, compute the Jacobi constant and the Z component of the angular momentum. Are these quantities constant?

c) Plot the six orbital elements as a function of time.

d) Identify features similar to and different from Fig. 2.3.5

Assigned Problems – Problem 3 (2/2)

e) Compare the node rate predicted by

$$\dot{\Omega} = -\frac{3}{2} J_2 \frac{n}{(1-e^2)^2} \left(\frac{a_e}{a}\right)^2 \cos i$$

with a value estimated from (c).

f) Compare the amplitude of the semimajor axis periodic term with

$$a(t) = \bar{a} + 3\bar{n}\bar{a}J_2 \left(\frac{a_e}{\bar{a}}\right)^2 \sin^2 \frac{\bar{\imath}(\cos(2\omega + 2M))}{2\dot{\omega}_s + 2\dot{M}_s}$$

g) Plot the ground track. Does the ground track repeat after one day?

a	e	i	Ω	ω	M_0
25500.0 km	0.0015	63 deg	-60 deg	0 deg	0 deg